

***-Polynomial Identities of Matrices with the Symplectic Involution: the Low Degrees.**

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1. Introduction.

Let A be an associative algebra with involution $*$ over a field F . Amitsur [2] defined a $*$ -polynomial to be a polynomial where the variables can also appear with a $*$ (such polynomials are then evaluated on associative algebras with involution in the obvious way). When $A = M_n(F)$, the $n \times n$ matrices with entries in F , the ideal $I(M_n(F), *)$ of $*$ -identities coincides with either $I(M_n(F), t)$ for t the transpose involution or $I(M_n(F), s)$ for s the symplectic involution (n even) [17, Thm. 3.1.62]. In both cases, the question of what is the minimal degree of a $*$ -polynomial identity is still open. In [6], Giambruno shows that over a field F of characteristic not 2, a $*$ -polynomial identity for $M_n(F)$ with $n > 2$ must be of degree greater than n . In the symplectic case, Drensky and Giambruno [5] improve on this by showing the minimal degree of a $*$ -polynomial identity for $M_n(F, s)$ with $n > 4$ to be greater than $n + 1$.

We also note that the $*$ -identities for $M_2(F)$, F of characteristic 0, have been completely determined by Levchenko, Drensky and Giambruno [4, 5, 9, 10] for both types of involutions. In [3], the present authors established the minimal degree for $*$ -identities of $M_n(F, t)$ when $n < 5$ and also provided generators for the minimal degree identities. The purpose of this paper is to do the same for $M_n(F, s)$ when $n < 5$. The general case $n \geq 5$ for $*$ = t or s will be treated separately.

As always, there are two aspects to this type of results: producing the identities and showing that there are no others. The existence results hold for arbitrary fields and are proved using the fact that matrix algebras or some subspaces are in some sense of degree 1 or 2. For the uniqueness results we must impose some mild characteristic restrictions and these proofs ultimately depend on substitutions. We begin with a few preliminaries.

We first recall that if the characteristic of the field F is not 2, the existence of a $*$ -identity of a given degree d in an algebra A is equivalent to the existence of a polynomial of the same degree which yields 0 for every

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substitution where the first k variables are replaced by symmetric elements of $(A, *)$ and the remaining $d - k$ variables are replaced by skew-symmetric elements of $(A, *)$, for k some integer between 0 and d (see for example [6]).

With this in mind, we remark that when $n = 2$, the 2×2 symmetric matrices $H_2(F, s) = FI$, I the identity matrix, and so $[x, y] := xy - yx$ vanishes whenever x is replaced by any element of $H_2(F, s)$ and y by any element of $M_2(F, s)$. One easily checks that $M_2(F, s)$ has no $*$ -identity of degree 1 and that all $*$ -identities of degree 2 come from the above. Consequently, since the symplectic involution occurs only when n is even, the remainder of this paper will be concerned with the case $n = 4$.

Consider $M_4(F, s)$, s the symplectic involution, i.e., the involution induced by

$$\begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}.$$

In this case the symmetric elements of $M_4(F)$,

$$H_4(F, s) = \begin{bmatrix} A & B \\ C & A^t \end{bmatrix}, \quad A \in M_2(F), \quad B, C \in K_2(F, t),$$

and the skew-symmetric elements,

$$K_4(F, s) = \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix}, \quad A \in M_2(F), \quad B, C \in H_2(F, t).$$

By an abuse of notation, we will write $p(x_1, x_2, \dots, x_n)$, $x_1, x_2, \dots, x_k \in H_4(F, s)$ and $x_{k+1}, \dots, x_n \in K_4(F, s)$ if x_1, x_2, \dots, x_k are replaced by elements of $H_4(F, s)$ and x_{k+1}, \dots, x_n by elements of $K_4(F, s)$, and we will say that x_1, x_2, \dots, x_k are *symmetric variables*, x_{k+1}, \dots, x_n are *skew-symmetric variables* and that $p(x_1, x_2, \dots, x_n)$ is of *type* k , $n - k$. If the polynomial $p(x_1, x_2, \dots, x_n)$ vanishes whenever $x_1, x_2, \dots, x_k \in H_4(F, s)$ and $x_{k+1}, \dots, x_n \in K_4(F, s)$, then, since $M_2(F)$ embeds in $H_4(F, s)$ and $K_4(F, s)$ via

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & A^t \end{bmatrix} \quad \text{and} \quad A \mapsto \begin{bmatrix} A & 0 \\ 0 & -A^t \end{bmatrix}, \quad A \in M_2(F),$$

p must be a polynomial identity of $M_2(F)$. The multilinear identities of $M_2(F)$ of degree 5 are well-known (for instance, see [14]). They form a vector space of dimension 29 over F . To cut the problem down still further we will consider the subspaces HV_5 and KV_5 of symmetric and skew-symmetric elements of V_5 with respect to ι the reversal involution, i.e., the unique involution of the free associative algebra fixing the generators. These spaces have dimensions 15 and 14 respectively. Using this notation, we can present our

main results as a table that lists the dimensions of the subspaces of multilinear *-identities of $M_4(F, s)$:

H	K	HV_5	KV_5	V_5
0	5	0	0	0
1	4	0	0	0
2	3	0	6	6
3	2	7	6	13
4	1	7	10	17
5	0	10	10	20

The first (respectively second) column indicates the number of symmetric (respectively skew-symmetric) variables while the third (respectively fourth) column lists the dimension of the corresponding space of ι -symmetric (respectively ι -skew-symmetric) multilinear identities. The fifth column is the sum of the entries of columns 3 and 4, that is, the dimension of the space of multilinear identities of degree 5 having the prescribed number of symmetric and skew-symmetric variables. For example the second to last line says that the total dimension of the space of multilinear identities of $M_4(F, s)$ of degree 5 involving 4 symmetric and 1 skew-symmetric variables is 17. Of these, the ι -symmetric polynomials form a 7-dimensional subspace and the ι -skew-symmetric polynomials form a 10-dimensional subspace.

One should not expect such a profusion of identities for higher degree algebras. Naturally, the richer the supply of identities, the greater the number of substitutions needed to circumscribe them.

2. The Identities.

In large part, our identities will follow from the fact that the Pfaffian is a quadratic form on $H_4(F)$ which is a Jordan algebra of degree 2 with respect to this quadratic form. Letting q denote the *Pfaffian*, $q(x, y)$ the bilinearization of q and $t(x) := q(x, 1)$, the *trace*, we have for $x, y, z \in H_4(F, s)$,

$$x^2 = t(x)x - q(x)1 \quad \text{and} \quad xyx =: yU_x = q(x, \bar{y})x - q(x)\bar{y}, \quad (1)$$

where $\bar{y} = t(y)1 - y$. These equations linearize to

$$\begin{aligned} xV_y &= x \circ y := xy + yx = t(x)y + t(y)x - q(x, y)1, \\ yU_{x,z} &= \{x, y, z\} := xyz + zyx = q(x, \bar{y})z + q(z, \bar{y})x - q(x, z)\bar{y}. \end{aligned} \quad (2)$$

The trace $t(x) = \frac{1}{2}T(x)$, where T is the usual trace of 4×4 matrices. In particular, $t(xy + yx) = T(xy)$ which we also write $2t(xy)$ even though strictly

speaking this does not make sense since xy need not be an element of $H_4(F)$. Applying t to the first equation of (2) yields

$$q(x, y) = t(x)t(y) - t(xy). \quad (3)$$

We will denote the usual commutator by square brackets, and a less common triple product by pointy brackets:

$$[x, y] := xy - yx, \quad \langle x, y, z \rangle := xyz - zyx.$$

We shall also need the Jordan derivation

$$aD_{b,c} := a[V_b, V_c] = \{a, b, c\} - \{a, c, b\} = [a, [b, c]]. \quad (4)$$

By (4) and (2),

$$\begin{aligned} [x, [x_1, x_2]] &= q(x, \bar{x}_1)x_2 + q(x_2, \bar{x}_1)x - q(x, x_2)\bar{x}_1 \\ &\quad - q(x, \bar{x}_2)x_1 - q(x_1, \bar{x}_2)x + q(x, x_1)\bar{x}_2 \\ &= q(x, \bar{x}_1)x_2 - q(x, x_2)\bar{x}_1 - q(x, \bar{x}_2)x_1 + q(x, x_1)\bar{x}_2, \end{aligned} \quad (5)$$

since $q(x_2, \bar{x}_1) = q(\bar{x}_2, x_1)$. It is then easy to check that

$$q(aD_{b,c}, d) = -q(a, dD_{b,c}) \quad \text{for } a, b, c, d \in H_4.$$

In particular, since $[H_4, H_4] = K_4$,

$$q([x_1, y], x_2) = -q(x_1, [x_2, y]), \quad x_i \in H_4, \quad y \in K_4. \quad (6)$$

Since $[x, y]$ is of trace 0,

$$\overline{[x, y]} = -[x, y] \quad \text{and} \quad [\bar{x}, y] = -[x, y]. \quad (7)$$

ι -Skew-Symmetric Identities.

We first obtain ι -skew-symmetric identities that are consequences of Wagner's central identity [13].

Lemma 2.1. *If $x_i \in H_4$, $y_j \in K_4$ and z is an arbitrary variable then*

$$[x_1, y_1]^2 \quad \text{and} \quad [x_1, y_1] \circ [x_2, y_2]$$

are central polynomials and hence

$$\begin{aligned} p_1(x_1, y_1, z) &= [[x_1, y_1]^2, z], \\ p_2(x_1, x_2, y_1, y_2, z) &= [[x_1, y_1] \circ [x_2, y_2], z] \end{aligned}$$

are identities of $M_4(F, s)$.

PROOF. Since $[x_i, y_i]$ is a symmetric element of trace 0, the first statement follows from (1) and (2). \square

Replacing z by y_3 , (respectively x_3) in p_2 and taking all possible permutations yields a 6-dimensional space of ι -skew-symmetric identities of type 2,3, (respectively 3,2).

Lemma 2.2. *If $x_i \in H_4$, then $[x_1, x_2]^2$ and hence its complete linearization*

$$[x_1, x_2] \circ [x_3, x_4] + [x_1, x_4] \circ [x_3, x_2]$$

are central polynomials and so for an arbitrary variable z ,

$$\begin{aligned} p_3(x_1, x_2, z) &= [[x_1, x_2]^2, z], \\ p_4(x_1, x_2, x_3, x_4, z) &= [[x_1, x_2] \circ [x_3, x_4] + [x_1, x_4] \circ [x_3, x_2], z] \end{aligned}$$

are identities of $M_4(F, s)$.

PROOF. It was noted by Jacobson [13] that while $[a, b] \in K_4$ when $a, b \in H_4$, the polynomial

$$[x_1, x_2]^2 = (x_1 U_{x_2}) \circ x_1 - x_1^2 U_{x_2} - x_2^2 U_{x_1}$$

is a Jordan polynomial which is a central identity for Jordan algebras of a nondegenerate quadratic form. Therefore, the same holds for its complete linearization. \square

For $x_i \in H_4$, the polynomial $p_4[x_1, x_2, x_3, x_4, x_5]$ along with those obtained by permuting the variables span a space of dimension 10 of ι -skew-symmetric identities of type 5,0. Moreover, for $y \in K_4$, the polynomials

$$\begin{aligned} p_4(x_1, x_2, x_3, x_4, y) &= [[x_1, x_2] \circ [x_3, x_4] + [x_1, x_4] \circ [x_3, x_2], y] \\ p_4(x_1, x_3, x_2, x_4, y) &= [[x_1, x_3] \circ [x_2, x_4] + [x_1, x_4] \circ [x_2, x_3], y] \end{aligned} \quad (8)$$

span a space of dimension 2 of ι -skew-symmetric identities of type 4,1.

Lemma 2.3. *If $x, y, z \in H_4$ and the trace $t(z) = 0$ then*

$$p_5(x, y, z) = [[x, y] \circ z, y] = x \circ (z U_y) - z \circ (x U_y) - (x \circ z) U_y + \{z x y^2\} \quad (9)$$

evaluates to 0 in H_4 .

PROOF. Expanding, one can check that equation (9) holds in any special Jordan algebra. Since $t(z) = 0$, $\bar{z} = -z$. Using equations (1), and (2),

$$\begin{aligned}
& x \circ (zU_y) - z \circ (xU_y) - (x \circ z)U_y + \{z \ x \ y^2\} \\
&= x \circ (q(y, \bar{z})y - q(y)\bar{z}) - z \circ (q(y, \bar{x})y - q(y)\bar{x}) \\
&\quad - (t(x)z - q(x, z)1)U_y + q(z, \bar{x})y^2 + q(y^2, \bar{x})z - q(y^2, z)\bar{x} \\
&= -q(y, z)y \circ x + q(y)x \circ z - q(y, \bar{x})y \circ z \\
&\quad + q(y)\bar{x} \circ z - t(x)zU_y + q(y^2, \bar{x})z - q(y^2, z)\bar{x}, \\
&\hspace{25em} \text{since } q(z, \bar{x}) = q(\bar{z}, x) \\
&= -q(y, z)(t(y)x + t(x)y - q(y, x)1) + 2t(x)q(y)z - q(y, \bar{x})(t(y)z \\
&\quad - q(y, z)1) - t(x)q(y, \bar{z})y + t(x)q(y)\bar{z} + q(t(y)y - q(y)1, \bar{x})z \\
&\quad - q(t(y)y - q(y)1, z)\bar{x}, \\
&\hspace{25em} \text{using } x + \bar{x} = t(x) \\
&= -q(y, z)t(y)x - q(y, z)t(x)y + q(y, z)q(y, x)1 + 2t(x)q(y)z \\
&\quad - q(y, \bar{x})t(y)z + q(y, \bar{x})q(y, z)1 + t(x)q(y, z)y - t(x)q(y)z \\
&\quad + t(y)q(y, \bar{x})z - q(y)t(\bar{x})z - t(y)q(y, z)\bar{x} + q(y)t(z)\bar{x}, \\
&\hspace{25em} \text{since } t(x) = t(\bar{x}) \text{ and } t(z) = 0 \\
&= -t(y)q(y, z)(x + \bar{x}) + t(y)t(x)q(y, z)1 + 2t(x)q(y)z - 2t(x)q(y)z \\
&= 0. \hspace{10em} \square
\end{aligned}$$

Replacing x by x_1 and linearizing y to x_2, x_3 in (9) yields

$$[[x_1, x_2] \circ z, x_3] + [[x_1, x_3] \circ z, x_2].$$

Finally replacing z by $[x_4, y]$ with $x_i \in H_4$ and $y \in K_4$ (so that $[x_4, y] \in H_4$ and is of trace 0) yields an ι -skew-symmetric 4,1 identity

$$p_6(x_1, x_2, x_3, x_4, y) = [[x_1, x_2] \circ [x_4, y], x_3] + [[x_1, x_3] \circ [x_4, y], x_2]. \quad (9')$$

Permuting the indices yields 12 such polynomials which span a subspace of dimension 8. These together with the two identities (8) span a 10-dimensional ι -skew-symmetric subspace of KV_5 .

Remark. We now have spaces of multilinear ι -skew-symmetric identities whose dimensions correspond to those in column 4 of the table. They are all consequences of the fact that H_4 is a Jordan algebra of degree 2.

ι -Symmetric Identities.

Lemma 2.4. *If $x_i \in H_4$ and $y_j \in K_4$ then*

$$\begin{aligned} r_1(x_1, x_2, x_3, y_1, y_2) &= x_3 D_{[x_1, y_2], [x_2, y_1]} \\ &+ \sum_{\sigma \in \mathcal{S}_2} (-1)^\sigma ([x_{\sigma(1)}, y_{\sigma(2)}], y_{\sigma(1)}) D_{x_3, x_{\sigma(2)}} \\ &+ [x_3, y_{\sigma(1)}] D_{[x_{\sigma(1)}, y_{\sigma(2)}], x_{\sigma(2)}} \end{aligned} \quad (10)$$

evaluates to 0 in M_4 .

PROOF. Computing each term separately using (5), (6) and (7),

$$\begin{aligned} x_3 D_{[x_1, y_2], [x_2, y_1]} &= q(x_3, \overline{[x_1, y_2]}) [x_2, y_1] - q(x_3, [x_2, y_1]) \overline{[x_1, y_2]} \\ &- q(x_3, \overline{[x_2, y_1]}) [x_1, y_2] + q(x_3, [x_1, y_2]) \overline{[x_2, y_1]} \\ &= 2(q(x_3, [x_2, y_1]) [x_1, y_2] - q(x_3, [x_1, y_2]) [x_2, y_1]). \end{aligned}$$

Also

$$\begin{aligned} [[x_1, y_2], y_1] D_{x_3, x_2} &= q([[x_1, y_2], y_1], \overline{x_3}) x_2 - q([[x_1, y_2], y_1], x_2) \overline{x_3} \\ &- q([[x_1, y_2], y_1], \overline{x_2}) x_3 + q([[x_1, y_2], y_1], x_3) \overline{x_2} \\ &= -q(x_1, [[x_3, y_1], y_2]) x_2 - q(x_1, [[x_2, y_1], y_2]) \overline{x_3} \\ &+ q(x_1, [[x_3, y_1], y_2]) \overline{x_2} + q(x_1, [[x_2, y_1], y_2]) x_3. \end{aligned}$$

Similarly

$$\begin{aligned} [[x_2, y_1], y_2] D_{x_3, x_1} &= -q(x_2, [[x_3, y_2], y_1]) x_1 - q(x_2, [[x_1, y_2], y_1]) \overline{x_3} \\ &+ q(x_2, [[x_3, y_2], y_1]) \overline{x_1} + q(x_2, [[x_1, y_2], y_1]) x_3, \\ [x_3, y_1] D_{[x_1, y_2], x_2} &= -2(q(x_3, [x_2, y_1]) [x_1, y_2] + q([x_3, y_1], [x_1, y_2]) x_2) \\ &+ q([x_3, y_1], [x_1, y_2]) t(x_2), \end{aligned}$$

and

$$\begin{aligned} [x_3, y_2] D_{[x_2, y_1], x_1} &= -2(q(x_3, [x_1, y_2]) [x_2, y_1] + q([x_3, y_2], [x_2, y_1]) x_1) \\ &+ q([x_3, y_2], [x_2, y_1]) t(x_1). \end{aligned}$$

The terms in $[x_1, y_2]$ cancel as do those in $[x_2, y_1]$. Since, by (6),

$$q(x_1, [[x_2, y_1], y_2]) = q([[x_1, y_2], y_1], x_2),$$

the terms in x_3 as well as those in $\overline{x_3}$ cancel. It remains to consider terms in x_1 and x_2 . In x_1 we have

$$\begin{aligned} & q(x_2, [[x_3, y_2], y_1]) x_1 - q(x_2, [[x_3, y_2], y_1]) \overline{x_1} \\ & + 2q([x_3, y_2], [x_2, y_1]) x_1 - q([x_3, y_2], [x_2, y_1]) t(x_1) \\ &= 2q(x_2, [[x_3, y_2], y_1]) x_1 - q(x_2, [[x_3, y_2], y_1]) t(x_1) \\ & - 2q([[x_3, y_2], y_1], x_2) x_1 + q([[x_3, y_2], y_1], x_2) t(x_1) \\ &= 0. \end{aligned}$$

Similarly the terms in x_2 cancel and $r_1(x_1, x_2, x_3, y_1, y_2)$ is an identity. \square

The polynomial r_1 is alternating in y_1, y_2 . Permuting the x 's yields 6 linearly independent ι -symmetric identities of type 3,2.

Lemma 2.5. *If $x_i \in H_4$ and $y \in K_4$ then*

$$r_2(x_1, x_2, x_3, y) = \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (\{x_{\sigma(1)}, [x_{\sigma(2)}, y], [x_{\sigma(3)}, y]\} + 2[x_{\sigma(1)}, y]^\circ \langle x_{\sigma(2)}, y, x_{\sigma(3)} \rangle) \quad (11)$$

evaluates to 0 in M_4 .

PROOF. We consider each summand separately:

$$\begin{aligned} & \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma \{x_{\sigma(1)}, [x_{\sigma(2)}, y], [x_{\sigma(3)}, y]\} \\ &= \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (q(x_{\sigma(1)}, \overline{[x_{\sigma(2)}, y]})[x_{\sigma(3)}, y] \\ & \quad + q([x_{\sigma(3)}, y], \overline{[x_{\sigma(2)}, y]})x_{\sigma(1)} \\ & \quad - q(x_{\sigma(1)}, [x_{\sigma(3)}, y])\overline{[x_{\sigma(2)}, y]}) \\ &= \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (-q(x_{\sigma(1)}, [x_{\sigma(2)}, y])[x_{\sigma(3)}, y] \\ & \quad - q([x_{\sigma(3)}, y], [x_{\sigma(2)}, y])x_{\sigma(1)} \\ & \quad + q(x_{\sigma(1)}, [x_{\sigma(3)}, y])[x_{\sigma(2)}, y]) \\ &= \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (-q(x_{\sigma(1)}, [x_{\sigma(2)}, y])[x_{\sigma(3)}, y] \\ & \quad + q(x_{\sigma(1)}, [x_{\sigma(3)}, y])[x_{\sigma(2)}, y]), \\ & \quad \text{since } q(\cdot, \cdot) \text{ is symmetric,} \\ &= \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma -2q(x_{\sigma(1)}, [x_{\sigma(2)}, y])[x_{\sigma(3)}, y]. \end{aligned}$$

On the other hand

$$\begin{aligned} & \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma [x_{\sigma(1)}, y]^\circ \langle x_{\sigma(2)}, y, x_{\sigma(3)} \rangle \\ &= \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (t([x_{\sigma(1)}, y]) \langle x_{\sigma(2)}, y, x_{\sigma(3)} \rangle \\ & \quad + t(\langle x_{\sigma(2)}, y, x_{\sigma(3)} \rangle)[x_{\sigma(1)}, y] \\ & \quad - q([x_{\sigma(1)}, y], \langle x_{\sigma(2)}, y, x_{\sigma(3)} \rangle)1) \\ &= \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma t(\langle x_{\sigma(2)}, y, x_{\sigma(3)} \rangle)[x_{\sigma(1)}, y], \end{aligned}$$

as the first term vanishes since $t([x_{\sigma(1)}, y]) = 0$ and the last term also vanishes since the matrix trace

$$\begin{aligned} T([x_1, y] < x_2, y, x_3 >) \\ &= T(x_1 y x_2 y x_3 - y x_1 x_2 y x_3 - x_1 y x_3 y x_2 + y x_1 x_3 y x_2) \\ &= T(x_1 y x_2 y x_3 - x_2 y x_3 y x_1 - x_1 y x_3 y x_2 + x_3 y x_2 y x_1), \end{aligned}$$

so the alternating sum will be 0. We are therefore left with

$$2 \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (-q(x_{\sigma(1)}, [x_{\sigma(2)}, y])[x_{\sigma(3)}, y] + t(< x_{\sigma(2)}, y, x_{\sigma(3)} >)[x_{\sigma(1)}, y]).$$

In that expression, the coefficient of $[x_3, y]$ is

$$\begin{aligned} 2(-q(x_1, [x_2, y]) + t(< x_1, y, x_2 >) + q(x_2, [x_1, y]) - t(< x_2, y, x_1 >)) \\ = 4(t(< x_1, y, x_2 >) - q(x_1, [x_2, y])). \end{aligned}$$

Since

$$4t(< x_1, y, x_2 >) = 2T(x_1 y x_2 - x_2 y x_1)$$

and

$$\begin{aligned} 4q(x_1, [x_2, y]) &= 2T(x_1, [x_2, y]) \\ &= 2T(x_1 x_2 y - x_1 y x_2), \end{aligned}$$

our polynomial evaluates to 0. \square

The identity r_2 is alternating in the x_i 's and its linearization in y together with the 6 polynomials obtained from $r_1(x_1, x_2, x_3, y_1, y_2)$ (of lemma 2.4) spans a 7-dimensional space of ι -symmetric identities of type 3,2.

Lemma 2.6. *If $x, x_i \in H_4$ and $y \in K_4$ then*

$$\begin{aligned} r_3(x_1, x_2, x, y) &= [[y, x], [x, [x_1, x_2]]] \\ &+ \sum_{\sigma \in \mathcal{S}_2} (-1)^\sigma ([\{x, [y, x], x_{\sigma(1)}\}, x_{\sigma(2)}] \\ &+ [x_{\sigma(1)}, [x_{\sigma(2)} U_x, y]] + [x_{\sigma(1)}, [y, x_{\sigma(2)}] U_x]) \end{aligned} \tag{12}$$

evaluates to 0 in M_4 .

PROOF. Recalling that $\overline{x_1} = t(x_1)1 - x_1$, (5) yields

$$\begin{aligned} [[y, x], [x, [x_1, x_2]]] &= q(x, \overline{x_1})[[y, x], x_2] + q(x, x_2)[[y, x], x_1 1] \\ &- q(x, \overline{x_2})[[y, x], x_1] - q(x, x_1)[[y, x], x_2]. \end{aligned} \tag{13}$$

Since $[y, x] \in H_2$ and $\overline{[y, x]} = -[y, x]$, using (2),

$$\{x, [y, x], x_1\} = -q(x, [y, x])x_1 - q(x_1, [y, x])x + q(x, x_1)[y, x]$$

and

$$\begin{aligned} & [\{x, [y, x], x_1\}, x_2] - [\{x, [y, x], x_2\}, x_1] \\ &= -q(x, [y, x])[x_1, x_2] - q(x_1, [y, x])[x, x_2] + q(x, x_1)[[y, x], x_2] \\ & \quad + q(x, [y, x])[x_2, x_1] + q(x_2, [y, x])[x, x_1] - q(x, x_2)[[y, x], x_1]. \end{aligned} \quad (14)$$

Since $[\overline{x_2}, y] = -[x_2, y]$,

$$\begin{aligned} & [x_1, [x_2 U_x, y]] - [x_2, [x_1 U_x, y]] = \\ & \quad q(x, \overline{x_2})[x_1, [x, y]] + q(x)[x_1, [x_2, y]] \\ & \quad - q(x, \overline{x_1})[x_2, [x, y]] - q(x)[x_2, [x_1, y]] \end{aligned} \quad (15)$$

and

$$\begin{aligned} & [x_1, [y, x_2] U_x] - [x_2, [y, x_1] U_x] = \\ & \quad -q(x, [y, x_2])[x_1, x] + q(x)[x_1, [y, x_2]] \\ & \quad + q(x, [y, x_1])[x_2, x] - q(x)[x_2, [y, x_1]]. \end{aligned} \quad (16)$$

Adding (13), (14), (15) and (16), we obtain

$$\begin{aligned} & -2q(x, [y, x])[x_1, x_2] + (q(x_1, [y, x]) + q(x, [y, x_1]))[x_2, x] \\ & \quad - (q(x_2, [y, x]) + q(x, [y, x_2]))[x_1, x] \end{aligned}$$

which is 0 by (6). □

Linearizing r_3 , replacing x by x_3, x_4 , yields an identity which is symmetric in x_3, x_4 and alternating in x_1, x_2 . Therefore permuting the variables produces 6 polynomials, and one can check that they are linearly independent. These 6 identities along with the identity from the next lemma will provide a 7-dimensional space of ι -symmetric identities of type 4,1.

Lemma 2.7. *If $x_i \in H_4$ and $y \in K_4$ then*

$$\begin{aligned} r_4(x_1, x_2, x_3, x_4, y) = & \sum_{\sigma \in \mathcal{S}_4} (-1)^\sigma \left(\frac{1}{2} [x_{\sigma(1)}, [[x_{\sigma(2)}, x_{\sigma(3)}], [x_{\sigma(4)}, y]] \right. \\ & \left. + \{[x_{\sigma(1)}, x_{\sigma(2)}], y, [x_{\sigma(3)}, x_{\sigma(4)}]\} \right) \end{aligned}$$

evaluates to 0 in M_4 .

PROOF. Consider first

$$\begin{aligned} & [[x_2, x_3], [x_4, y]] \\ &= -q([x_4, y], \overline{x_2})x_3 + q([x_4, y], x_3)\overline{x_2} + q([x_4, y], \overline{x_3})x_2 - q([x_4, y], x_2)\overline{x_3} \\ &= q([x_4, y], x_2)x_3 + q([x_4, y], x_3)\overline{x_2} - q([x_4, y], x_3)x_2 - q([x_4, y], x_2)\overline{x_3}. \end{aligned}$$

So

$$[x_1, [[x_2, x_3], [x_4, y]]] = 2(q([x_4, y], x_2)[x_1, x_3] - q([x_4, y], x_3)[x_1, x_2])$$

and the alternating sum

$$\begin{aligned} & \sum_{\sigma \in \mathcal{S}_4} (-1)^\sigma \frac{1}{2} [x_{\sigma(1)}, [[x_{\sigma(2)}, x_{\sigma(3)}], [x_{\sigma(4)}, y]] \\ &= \sum_{\sigma \in \mathcal{S}_4} (-1)^\sigma 2q([x_{\sigma(3)}, y], x_{\sigma(4)})[x_{\sigma(1)}, x_{\sigma(2)}]. \quad (17) \end{aligned}$$

Assume that we are working over an algebraically closed field of characteristic not 2. Let $\chi_s(z)$ be the characteristic polynomial of a skew-symmetric element s and $\delta(s)$ its discriminant. If $\delta(s) \neq 0$ then s is diagonalizable and the four distinct eigenvalues come in pairs $\gamma_1, -\gamma_1, \gamma_2, -\gamma_2$. If $c \in M_4$ such that $csc^{-1} = d = \text{diag}\{\gamma_1, \gamma_2, -\gamma_1, -\gamma_2\}$ then $cs = dc$; applying the involution $-sc^* = c^*(-d)$ and $dcc^* = cc^*d$. Since the entries of d are distinct, cc^* is diagonal and, replacing c by $(cc^*)^{-\frac{1}{2}}c$ if necessary, we may assume that cc^* is the identity matrix and that conjugation by c stabilizes H_4 and K_4 . Since the elements $s \in K_4$ with $\delta(s) \neq 0$ are Zariski dense, it suffices to check that $r_4(x_1, x_2, x_3, x_4, y)$ vanishes when $y = d$, a skew-symmetric diagonal element.

The following basis of H_4 :

$$\begin{aligned} e_1 &= e_{11} + e_{33}, \quad e_2 = e_{22} + e_{44}, \quad h_{12} = -(e_{12} + e_{43}), \quad h_{21} = e_{21} + e_{34} \\ h_{14} &= e_{14} - e_{23}, \quad h_{32} = e_{32} - e_{41} \end{aligned}$$

consists of hyperbolic pairs with respect to the Pfaffian q . Since every symmetric entry x_i of $r_4(x_1, x_2, x_3, x_4, y)$ is in a commutator, replacing x_i by the identity matrix causes the polynomial to vanish. Since $r_4(x_1, x_2, x_3, x_4, y)$ is alternating in x_1, x_2, x_3, x_4 , we therefore need only check that a vanishes for the five substitutions of four distinct elements chosen from $e_1, h_{12}, h_{21}, h_{14}, h_{32}$. For these elements, $[x, d]$ is a multiple of x . So for $q(x_i, [x_j, d])$ to be nonzero, the pair (x_i, x_j) must be hyperbolic. Of the five substitutions above only $h_{12}, h_{21}, h_{14}, h_{32}$ has two hyperbolic pairs, all the others have exactly one. We will compute this substitution in detail. The others are similar and easier.

In any case $q([h_{12}, d], h_{21}) = \gamma_2 - \gamma_1$, $q([h_{14}, d], h_{32}) = -(\gamma_1 + \gamma_2)$, $[h_{12}, h_{21}] = \text{diag}\{-1, 1, 1, -1\}$ and $[h_{14}, h_{32}] = \text{diag}\{-1, -1, 1, 1\}$. If we let $x_1 = h_{12}$, $x_2 = h_{21}$, $x_3 = h_{14}$ and $x_4 = h_{32}$, the eight permutations yielding a nonzero result in (17) do not change the sign of the expression,

$$\begin{aligned} & \sum_{\sigma \in \mathcal{S}_4} (-1)^\sigma 2q([x_{\sigma(3)}, d], x_{\sigma(4)})[x_{\sigma(1)}, x_{\sigma(2)}] \\ &= 8((\gamma_2 - \gamma_1) \text{diag}\{-1, -1, 1, 1\} - (\gamma_1 + \gamma_2) \text{diag}\{-1, 1, 1, -1\}) \\ &= 16 \text{diag}\{\gamma_1, -\gamma_2, -\gamma_1, \gamma_2\}. \end{aligned}$$

Consider the second summand of $r_4(h_{12}, h_{21}, h_{14}, h_{32}, d)$. By the skew-symmetry of the commutator and the symmetry of the triple product in the two exterior entries, we need only compute three triple products:

$$\begin{aligned} & \{[h_{12}, h_{21}], d, [h_{14}, h_{32}]\} = 2 \text{diag}\{\gamma_1, -\gamma_2, -\gamma_1, \gamma_2\}, \\ & -\{[h_{12}, h_{14}], d, [h_{21}, h_{32}]\} = -4 \text{diag}\{\gamma_1, 0, -\gamma_1, 0\}, \\ & \{[h_{12}, h_{32}], d, [h_{14}, h_{21}]\} = 4 \text{diag}\{0, \gamma_2, 0, -\gamma_2\}. \end{aligned}$$

$$\begin{aligned} & \sum_{\sigma \in \mathcal{S}_4} (-1)^\sigma \{[x_{\sigma(1)}, x_{\sigma(2)}], y, [x_{\sigma(3)}, x_{\sigma(4)}]\} = 8(2 \text{diag}\{\gamma_1, -\gamma_2, -\gamma_1, \gamma_2\} \\ & \quad - 4 \text{diag}\{\gamma_1, 0, -\gamma_1, 0\} + 4 \text{diag}\{0, \gamma_2, 0, -\gamma_2\}) \\ & \quad = -16 \text{diag}\{\gamma_1, -\gamma_2, -\gamma_1, \gamma_2\} \end{aligned}$$

and $r_4(h_{12}, h_{21}, h_{14}, h_{32}, d) = 0$. □

The above proof is unsatisfactory in that it sheds no light on the reasons why $r_4(x_1, x_2, x_3, x_4, y)$ is an identity.

In [14] the second author showed that Jordan algebras of degree 2 satisfy the following identity:

$$r_5(x_1, x_2, x_3; y) := y^2 \mathbb{S}_3(\mathbb{V}_{x_1}, \mathbb{V}_{x_2}, \mathbb{V}_{x_3}) - (y \mathbb{S}_3(\mathbb{V}_{x_1}, \mathbb{V}_{x_2}, \mathbb{V}_{x_3})) \circ y$$

(where \mathbb{S}_3 is the standard polynomial of degree 3). This can be rewritten in associative terms

$$r_5(x_1, x_2, x_3; y) = \mathbb{S}_4(x_1, x_2, x_3, y^2) - y \circ \mathbb{S}_4(x_1, x_2, x_3, y),$$

or in terms of linear triple products

$$\begin{aligned} r_5(x_1, x_2, x_3; y) := & \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (\{ \{y \ y \ x_{\sigma(1)}\} \ x_{\sigma(2)} \ x_{\sigma(3)} \} \\ & - \{ \{y \ x_{\sigma(1)} \ x_{\sigma(2)}\} \ x_{\sigma(3)} \ y \} \\ & - \{ x_{\sigma(1)} \ \{y \ x_{\sigma(2)} \ x_{\sigma(3)}\} \ y \}), \end{aligned}$$

and we keep the same notation for the linearization of r_5 :

$$r_5(x_1, x_2, x_3, x, y) := (x \circ y) S_3(V_{x_1}, V_{x_2}, V_{x_3}) - (x S_3(V_{x_1}, V_{x_2}, V_{x_3})) \circ y \\ - (y S_3(V_{x_1}, V_{x_2}, V_{x_3})) \circ x.$$

The ten polynomials

$$r_5^{ij} := r_5(x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_5, x_i, x_j), \quad 1 \leq i < j \leq 5 \quad (18)$$

are linearly independent and, since $H_4(F)$ is a Jordan algebra of degree 2, they are ι -symmetric identities of type 5,0.

Remark. As was noted above, permuting the variables in p_4 (from Lemma 2.2) and r_5 will provide generators for $I(H_4)$ as a 20-dimensional $F[\mathcal{S}_5]$ -module. One can check (under mild characteristic restrictions on F) that the character of this representation is $[3,2] + [3,1,1] + [2,2,1] + [2,1,1,1]$ in the usual notation for Young diagrams (where $[3,2]$ and $[2,2,1]$ are generated by p_4 while $[3,1,1]$ and $[2,1,1,1]$ are generated by r_5).

We have now produced spaces of identities of $M_4(F, s)$ corresponding to all the dimensions in our table. Assuming some restrictions on the base field F , we now proceed to show that all $*$ -identities of (minimal) degree 5 of $M_4(F, s)$ are consequences of the identities presented thus far.

3. Uniqueness.

We recall that whenever the degree of each variable x_i in an identity $p(x_1, \dots, x_m)$ is less than the cardinality $|F|$ of F , then each homogeneous component of p is also an identity. Accordingly, we will carry out a case by case study of homogeneous $*$ -PI's. In order to simplify our calculations, we first recall a useful result of Ma.

In [11], Ma defines a partial ordering on the homogeneous elements of the free associative algebra $F[X]$. A polynomial $p(x_1, \dots, x_m)$ is said to be of type $[n_1, \dots, n_m]$ if the n_j 's are the degrees of the x_i 's rearranged in nonincreasing order. (If an integer is repeated k times, we denote this with an exponent; for example, $[3, 2^2, 1^3]$ means $[3, 2, 2, 1, 1, 1]$.) Given homogeneous polynomials p and p' of degree n and n' and of type $[n_1, \dots, n_m]$ and $[n'_1, \dots, n'_m]$ respectively, we say that p is lower than p' in the partial ordering if and only if either (i) $n < n'$, or (ii) $n = n'$ and $n_j > n'_j$ for the first j such that $n_j \neq n'_j$. Otherwise two polynomials are not comparable.

Now let $f(x_1, \dots, x_r, y_1, \dots, y_s, \dots, z_1, \dots, z_t)$ be a homogeneous polynomial identity of type $[m^r, n^s, \dots, u^t]$ on some subspace V of $M_n(F)$ (e.g. $V = H_n$ or K_n) and set $m_0 = \max\{m, n, \dots, u, r, s, \dots, t\}$. The following

theorem provides a relation of symmetry between variables of equal degree in f , depending on how many there are.

Theorem 3.1. *If the characteristic of F does not divide $m_0!$ and $|F| > 2m_0 - 1$, then $f = f_0 + f_1$ where $f_i, i = 0, 1$, are identities of the same type as f and for each k with $0 < k \leq m_0$, f_0 is symmetric or skew-symmetric in all variables of degree k depending on whether k is even or odd while f_1 comes from identities of lower type. \square*

This result extends to $*$ -identities of degree n as long as we keep the k symmetric variables and $n - k$ skew-symmetric variables apart. By assuming the symmetry properties of Theorem 3.1 among variables, we can greatly reduce the number of arbitrary coefficients involved in our calculations. We first establish that there are no identities with only skew-symmetric variables.

Theorem 3.2. $K_4(F, s)$ has no multilinear identity of degree 5.

PROOF. If $f(y_1, y_2, y_3, y_4, y_5)$ were such an identity, one could assume that it contains the monomial $y_1 y_2 y_3 y_4 y_5$ (by relabeling the variables) with coefficient 1; we then compute

$$f(e_{13}, e_{31}, e_{12} - e_{43}, e_{24}, e_{42}) = e_{12} + \sum_{ij \neq 12} \lambda_{ij} e_{ij},$$

which shows that f could not possibly vanish. \square

We now show that there are no $*$ -identities of type 1,4.

Theorem 3.3. $M_4(F, s)$ has no multilinear identity $f(x, y_1, y_2, y_3, y_4)$ of degree 5 with $x \in H_4(F, s)$, $y_i \in K_4(F, s)$.

PROOF. By Theorem 3.1, we may assume that such an identity f is alternating in its skew variables y_1, \dots, y_4 ; so we begin with $f(x, y_1, y_2, y_3, y_4) = B_1 + B_2 + B_3 + B_4 + B_5$ where

$$B_i = \sum_{\sigma \in \mathcal{S}_4} \alpha_\sigma^i y_{\sigma(1)} \cdots y_{\sigma(i-1)} x y_{\sigma(i)} \cdots y_{\sigma(4)}.$$

If f contains a term $\alpha_\sigma^3 y_{\sigma(1)} y_{\sigma(2)} x y_{\sigma(3)} y_{\sigma(4)}$, then assume that $\sigma = 1$ and get $f(e_{12} + e_{43}, e_{13}, e_{31}, e_{24}, e_{42}) = \alpha_1^3 e_{12} + \dots$; hence $\alpha_\sigma^3 = 0$ and $f = B_1 + B_2 + B_4 + B_5$. Next,

$$f(e_{11} + e_{33}, e_{13}, e_{31}, e_{12} - e_{43}, e_{24}) = (\alpha_\sigma^1 + \alpha_\sigma^2) e_{14} + \sum_{ij \neq 14} \lambda_{ij} e_{ij},$$

which implies $\alpha_\sigma^2 = -\alpha_\sigma^1$ and hence $B_1 = -B_2$. Similarly, the transpose of this substitution, namely $f(e_{11} + e_{33}, e_{42}, e_{34} - e_{21}, e_{13}, e_{31})$, will yield $\alpha_\sigma^4 = -\alpha_\sigma^5$ and $B_4 = -B_5$. Adding to this the substitution

$$f(e_{21} + e_{34}, e_{13}, e_{42}, e_{14} + e_{23}, e_{31}) = (\alpha_\sigma^2 + \alpha_{\sigma(24)}^5) e_{11} + \sum_{ij \neq 11} \lambda_{ij} e_{ij},$$

which says that $\alpha_\sigma^2 = -\alpha_{\sigma(24)}^5$, we reduce the polynomial f to

$$f = \sum_{\sigma \in \mathcal{S}_4} \alpha_\sigma [y_{\sigma(1)}, x] y_{\sigma(2)} y_{\sigma(3)} y_{\sigma(4)} + \beta_\sigma y_{\sigma(1)} y_{\sigma(2)} y_{\sigma(3)} [y_{\sigma(4)}, x],$$

where $\alpha_\sigma = -\beta_{\sigma(24)}$. So we will get $f \equiv 0$ as soon as $\alpha_\sigma = 0$ for all σ .

The coefficient of e_{43} in $f(e_{11} + e_{33}, e_{12} - e_{43}, e_{11} - e_{33}, e_{31}, e_{13})$ is

$$\alpha_\sigma - \alpha_{\sigma(23)} + \alpha_{\sigma(234)} = 0. \quad (19)$$

Multiplication on the right by the permutation (34) provides the relation $\alpha_{\sigma(34)} - \alpha_{\sigma(234)} + \alpha_{\sigma(23)} = 0$ which when added to equation (19) yields $\alpha_\sigma = -\alpha_{\sigma(34)}$, while multiplication on the right by the permutation (24) yields

$$\alpha_{\sigma(24)} - \alpha_{\sigma(243)} + \alpha_{\sigma(34)} = 0. \quad (20)$$

Next, the coefficient of e_{12} in $f(e_{12} + e_{43}, e_{11} - e_{33}, e_{24}, e_{42}, e_{22} - e_{44})$ is

$$\alpha_\sigma + \alpha_{\sigma(243)} - \alpha_{\sigma(34)} = 0, \quad (21)$$

and adding (20) and (21) leads to $\alpha_\sigma = -\alpha_{\sigma(24)}$; we multiply this last relation on the right by (34) to deduce that $\alpha_{\sigma(34)} = -\alpha_{\sigma(243)}$, and we can now write (21) as $\alpha_\sigma + \alpha_{\sigma(243)} - \alpha_{\sigma(34)} = 3\alpha_\sigma = 0$ which implies $\alpha_\sigma = 0$ provided the characteristic of F is not 3. \square

We still have to establish uniqueness for identities of types 2,3, 3,2, 4,1, and 5,0. Each type is handled by way of a long list of substitutions and relations among coefficients. Although Theorem 3.1 helps us reduce the work considerably by providing some symmetry relationships among variables of the same degree, each type remains plagued by tedious calculations and a long litany of cases. We will spare the reader and present the detailed substitutions showing uniqueness for type 5,0 (i.e., identities of $H_4(F)$) and only briefly outline the outcomes for the other three types. A more detailed account of the substitutions for the latter three types can be provided upon request.

Theorem 3.4. *Under the hypotheses of Theorem 3.1 on the field F , any polynomial identity of $H_4(F, s)$ of minimal degree is a consequence of*

$$\begin{aligned} p_4(x_1, x_2, x_3, x_4, x_5) &= [[x_1, x_2] \circ [x_3, x_4] + [x_1, x_4] \circ [x_3, x_2], x_5], \\ r_5(x_1, x_2, x_3, x_4, x_5) &= (x_4 \circ x_5) S_3(\mathbf{V}_{x_1}, \mathbf{V}_{x_2}, \mathbf{V}_{x_3}) \\ &\quad - (x_4 S_3(\mathbf{V}_{x_1}, \mathbf{V}_{x_2}, \mathbf{V}_{x_3})) \circ x_5 \\ &\quad - (x_5 S_3(\mathbf{V}_{x_1}, \mathbf{V}_{x_2}, \mathbf{V}_{x_3})) \circ x_4. \end{aligned}$$

PROOF. Using the usual Young diagram notation, a homogeneous identity of degree 5 must be of type [5], [4, 1], [3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1] or [1, 1, 1, 1, 1]. We can readily dismiss the extreme cases. For [5], $f(x) = \alpha x^5$ clearly does not vanish on H_4 since for instance $f(e_{11} + e_{33}) = \alpha(e_{11} + e_{33})$. At the other extreme for [1, 1, 1, 1, 1], we assume by Theorem 3.1 that $f(x_1, x_2, x_3, x_4, x_5) = S_5$, and $S_5 \notin I(H_4, s)$ since the minimal degree of a standard identity for $H_n(F, s)$ is $2n - 2$ (see [18]).

Suppose now that $f(x, y) = \alpha_1 x^4 y + \alpha_2 x^3 y x + \alpha_3 x^2 y x^2 + \alpha_4 x y x^3 + \alpha_5 y x^4$ is a typical homogeneous identity of type [4, 1]. Then

$$f(e_{11} + e_{33}, e_{12} + e_{43}) = \alpha_1 e_{12} + \alpha_5 e_{43} = 0 \implies \alpha_1 = \alpha_5 = 0,$$

so $f(x, y) = \alpha_2 x^3 y x + \alpha_3 x^2 y x^2 + \alpha_4 x y x^3$. Next, we compute

$$\begin{aligned} f(e_{11} + e_{33} + e_{12} + e_{43}, e_{21} + e_{34}) &= (\alpha_2 + \alpha_3 + \alpha_4)(e_{332} + e_{43}) + \alpha_4 e_{11} = 0 \\ \implies \alpha_4 &= 0, \quad \alpha_2 = -\alpha_3 \end{aligned}$$

and hence $f(x, y) = \alpha(x^3 y x - x^2 y x^2)$. Finally, for $I =$ identity matrix, $f(I + e_{14} - e_{23}, e_{22} + e_{44}) = \alpha(e_{14} + e_{23}) = 0$ yields $\alpha = 0$. Therefore, there are no identities of type [4, 1].

The case [3, 2] requires four substitutions. We begin with a typical polynomial of this type:

$$\begin{aligned} f(x, y) &= \alpha_1 x^3 y^2 + \alpha_2 x^2 y x y + \alpha_3 x^2 y^2 x + \alpha_4 x y x y x + \alpha_5 x y^2 x^2 \\ &\quad + \alpha_6 y x y x^2 + \alpha_7 y^2 x^3 + \alpha_8 x y x^2 y + \alpha_9 y x^2 y x + \alpha_{10} y x^3 y. \end{aligned}$$

We successively compute

- $f(e_{11} + e_{33} + e_{12} + e_{43}, e_{21} + e_{34}) = (\alpha_2 + \alpha_4 + \alpha_8)e_{11} + \alpha_4(e_{12} + e_{43})$
 $+ (\alpha_6 + \alpha_9 + \alpha_{10})e_{21} + (\alpha_6 + \alpha_9)e_{22} + (\alpha_4 + \alpha_6 + \alpha_9)e_{33}$
 $+ (\alpha_2 + \alpha_8 + \alpha_{10})e_{34} + (\alpha_2 + \alpha_8)e_{44}$
 $\implies \alpha_4 = \alpha_{10} = 0, \quad \alpha_6 = -\alpha_9, \quad \alpha_2 = -\alpha_8,$

$$\begin{aligned} \therefore f(x, y) &= \alpha_1 x^3 y^2 + \alpha_2(x^2 y x y - x y x^2 y) + \alpha_3 x^2 y^2 x + \alpha_5 x y^2 x^2 \\ &\quad + \alpha_6(y x y x^2 - y x^2 y x) + \alpha_7 y^2 x^3; \end{aligned}$$

- $f(e_{11} + e_{33} + e_{14} - e_{23}, e_{22} + e_{44}) = \alpha_1 e_{14} - \alpha_7 e_{23}$
 $\implies \alpha_1 = \alpha_7 = 0,$

$$\begin{aligned} \therefore f(x, y) &= \alpha_2(x^2 y x y - x y x^2 y) + \alpha_3 x^2 y^2 x + \alpha_5 x y^2 x^2 \\ &\quad + \alpha_6(y x y x^2 - y x^2 y x); \end{aligned}$$

- $f(e_{11} + e_{33} + e_{12} + e_{43}, e_{22} + e_{44} + e_{21} - e_{34})$
 $= (\alpha_3 + \alpha_5)(e_{11} + e_{22} + e_{33} + e_{43}) \implies \alpha_5 = -\alpha_3,$
 $\therefore f(x, y) = \alpha_2(x^2yxy - xyx^2y) + \alpha_3(x^2y^2x - xy^2x^2)$
 $+ \alpha_6(yxyx^2 - yx^2yx);$
- $f(I + e_{14} - e_{23}, I + e_{32} - e_{41}) = (\alpha_6 - \alpha_2 - 2\alpha_3)(e_{11} + e_{22} - e_{33} - e_{44})$
 $+ (\alpha_2 + \alpha_6)(e_{32} - e_{41}) \implies \alpha_2 = -\alpha_3 = -\alpha_6,$
 $\therefore f(x, y) = \alpha(x^2yxy - xyx^2y - x^2y^2x + xy^2x^2 - yxyx^2 + yx^2yx)$
 $= \alpha[[x, y]^2, x] = \left(\frac{\alpha}{4}\right) p_4(x, y, x, y, x) = \alpha p_3(x, y, x).$

Thus, f is a consequence of p_4 . We move on to the case $[3, 1, 1]$.

By Theorem 3.1, a generic polynomial $f(x, y, z)$ of this type can be assumed to be skew-symmetric in its linear variables y and z :

$$f(x, y, z) = \alpha_1 x^3(yz - zy) + \alpha_2 x^2(yz - zy)x + \alpha_3 x(yz - zy)x^2$$

$$+ \alpha_4(yz - zy)x^3 + \alpha_5 x^2(yxz - zxy) + \alpha_6 x(yxz - zxy)x$$

$$+ \alpha_7(yxz - zxy)x^2 + \alpha_8 x(yx^2z - zx^2y) + \alpha_9(yx^2z - zx^2y)x$$

$$+ \alpha_{10}(yx^3z - zx^3y).$$

We first perform the three substitutions

- $f(e_{11} + e_{33}, e_{22} + e_{44}, e_{12} + e_{43}) = -\alpha_1 e_{12} + \alpha_4 e_{43} \implies \alpha_1 = \alpha_4 = 0,$
- $f(e_{11} + e_{33}, I, e_{12} + e_{43}) = (\alpha_5 + \alpha_8 + \alpha_{10})e_{12} - (\alpha_7 + \alpha_9 + \alpha_{10})e_{43}$
 $\implies \alpha_{10} = -\alpha_5 - \alpha_8 = -\alpha_7 - \alpha_9,$
- $f(e_{11} + e_{33}, e_{12} + e_{43}, e_{21} + e_{34}) = (\alpha_2 + \alpha_3)(e_{11} - e_{33})$
 $+ (\alpha_5 + \alpha_8)(e_{22} - e_{44}) \implies \alpha_5 = -\alpha_8, \quad \alpha_2 = -\alpha_3.$

From these, we further obtain $\alpha_{10} = 0$ and $\alpha_7 = -\alpha_9$, and we reduce f to

$$f(x, y, z) = \alpha_2(x^2(yz - zy)x - x(yz - zy)x^2)$$

$$+ \alpha_5(x^2(yxz - zxy) - x(yx^2z - zx^2y))$$

$$+ \alpha_6 x(yxz - zxy)x + \alpha_7((yxz - zxy)x^2 - (yx^2z - zx^2y)x).$$

We eliminate one more coefficient using the substitution

$$f(I, e_{12} + e_{43}, e_{11} + e_{33}) = \alpha_6(e_{43} - e_{12}) \implies \alpha_6 = 0.$$

Next, by setting $x = z$ and using the fact that there are no identities of type [4, 1], we obtain

$$\begin{aligned} f(x, y, x) &= (\alpha_7 - \alpha_2)x^3yx + (\alpha_5 + 2\alpha_2 - \alpha_7)x^2yx^2 + (-\alpha_2 - \alpha_5)xyx^3 \\ &\implies \alpha_2 = \alpha_7 = -\alpha_5, \end{aligned}$$

$$\begin{aligned} \therefore f(x, y, z) &= \alpha(x^2[y, z]x - x[y, z]x^2 - x^2(yxz - zxy) \\ &\quad + x(yx^2z - zx^2y) + (yxz - zxy)x^2 - (yx^2z - zx^2y)x \\ &= \left(-\frac{\alpha}{2}\right) r_5(y, z, x, x, x), \end{aligned}$$

and f is therefore a consequence of r_5 .

The type [2, 2, 1] involves 30 monomials. Assuming a typical polynomial $f(x, y, z)$ to be symmetric in its two quadratic variables x and y , we write

$$\begin{aligned} f(x, y, z) &= \alpha_1(x^2y^2 + y^2x^2)z + \alpha_2(xyxy + yxyx)z + \alpha_3(xy^2x + yx^2y)z \\ &\quad + \alpha_4(x^2zy^2 + y^2zx^2) + \alpha_5(xyzyx + yxzxy) + \alpha_6(xyzxy + yxzyx) \\ &\quad + \alpha_7z(x^2y^2 + y^2x^2) + \alpha_8z(xyxy + yxyx) + \alpha_9z(xy^2x + yx^2y) \\ &\quad + \alpha_{10}(xzx^2y^2 + yzyx^2) + \alpha_{11}(xzyxy + yzxyx) + \alpha_{12}(xzy^2x + yzx^2y) \\ &\quad + \alpha_{13}(x^2yzy + y^2xzx) + \alpha_{14}(xyxzy + yxyzx) + \alpha_{15}(yx^2zy + xy^2zx). \end{aligned}$$

By subtracting a multiple of $p_4(x, x, y, y, z)$ (respectively $p_4(x, x, z, y, y)$), we may assume that $\alpha_2 = 0$ (respectively $\alpha_{11} = 0$). We now perform the following substitutions:

$$f(e_{11} + e_{33}, e_{22} + e_{44}, e_{12} + e_{43}) = \alpha_4(e_{12} + e_{43}) \implies \alpha_4 = 0, \quad (22)$$

$$f(e_{12} + e_{43}, e_{21} + e_{34}, e_{12} + e_{43}) = \alpha_5(e_{12} + e_{43}) \implies \alpha_5 = 0, \quad (23)$$

$$\begin{aligned} f(e_{22} + e_{44}, e_{22} + e_{44}, e_{12} + e_{43}) &= 2\alpha_7e_{12} + (2\alpha_1 + 2\alpha_3)e_{43} \\ &\implies \alpha_7 = 0, \quad \alpha_1 = -\alpha_3, \end{aligned} \quad (24)$$

$$\begin{aligned} f(I, e_{11} + e_{33}, e_{21} + e_{34}) &= (\alpha_{10} + \alpha_{12})e_{21} + (\alpha_{13} + \alpha_{14} + \alpha_{15})e_{34} \\ &\implies \alpha_{10} = -\alpha_{12}, \quad \alpha_{13} + \alpha_{14} + \alpha_{15} = 0, \end{aligned} \quad (25)$$

$$\begin{aligned} f(e_{12} + e_{43}, I, e_{21} + e_{34}) &= (\alpha_{10} + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} \\ &\quad + 2\alpha_6)(e_{12} + e_{43}) \implies \alpha_6 = 0. \end{aligned} \quad (26)$$

We simplify f using (22) – (26) and obtain

$$\begin{aligned} f(x, y, z) &= \alpha_1((x^2y^2 + y^2x^2)z - (xy^2x + yx^2y)z) + \alpha_8z(xyxy + yxyx) \\ &\quad + \alpha_9z(xy^2x + yx^2y) + \alpha_{10}((xzx^2y^2 + yzyx^2) - (xzy^2x + yzx^2y)) \\ &\quad + \alpha_{13}(x^2yzy + y^2xzx) + \alpha_{14}(xyxzy + yxyzx) + \alpha_{15}(yx^2zy + xy^2zx). \end{aligned}$$

Once again using the fact that there are no identities of type [4, 1], we get

$$f(x, x, z) = 2(\alpha_8 + \alpha_9)zx^4 + 2(\alpha_{13} + \alpha_{14} + \alpha_{15})x^3zx \implies \alpha_8 = -\alpha_9. \quad (27)$$

Next,

$$f(e_{12} + e_{43}, e_{21} + e_{34}, I) = (\alpha_8 + \alpha_{14})I \implies \alpha_8 = -\alpha_{14}, \quad (28)$$

$$\begin{aligned} f(e_{22} + e_{44} + e_{14} - e_{23}, e_{11} + e_{33}, e_{11} + e_{33} + e_{32} - e_{41}) \\ = -(\alpha_1 + \alpha_{15})(e_{11} + e_{22}) + (2\alpha_{10} - \alpha_{13})e_{14} \\ + \alpha_{10}(e_{33} + e_{44}) + (-\alpha_1 + \alpha_{10} - \alpha_{13} + \alpha_{15})e_{23} \\ \implies \alpha_{10} = \alpha_{13} = \alpha_{15} = \alpha_1 = 0. \end{aligned} \quad (29)$$

Combining (29) and (25) yields $\alpha_{12} = \alpha_{14} = 0$, (24) gives us $\alpha_3 = 0$, and both (27) and (28) together imply $\alpha_8 = \alpha_9 = 0$. Thus, f is a consequence of p_4 .

The last case, type [2, 1, 1, 1] presents the most daunting computational task because it involves 60 monomials. Assuming $f(x_1, x_2, x_3, y)$ to be skew-symmetric in its variables x_1, x_2, x_3 of degree 1 (by 3.1), we only have 10 coefficients:

$$\begin{aligned} f(x_1, x_2, x_3, y) = & \alpha_1 y^2 \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \right) \\ & + \alpha_2 y \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} y x_{\sigma(2)} x_{\sigma(3)} \right) + \alpha_3 y \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} y x_{\sigma(3)} \right) \\ & + \alpha_4 y \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \right) y + \alpha_5 \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} y^2 x_{\sigma(2)} x_{\sigma(3)} \right) \\ & + \alpha_6 \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} y x_{\sigma(2)} y x_{\sigma(3)} \right) + \alpha_7 \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} y x_{\sigma(2)} x_{\sigma(3)} \right) y \\ & + \alpha_8 \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} y^2 x_{\sigma(3)} \right) + \alpha_9 \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} y x_{\sigma(3)} \right) y \\ & + \alpha_{10} \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \right) y^2. \end{aligned}$$

In $f(e_{12} + e_{43}, e_{21} + e_{34}, e_{32} - e_{41}, e_{11} + e_{33})$, the coefficients of e_{32} and e_{41} provide the relations

$$\alpha_1 + \alpha_3 + \alpha_8 = 0, \quad (30)$$

$$\alpha_5 + \alpha_7 + \alpha_{10} = 0. \quad (31)$$

We carry out the substitution $f(e_{14} - e_{23}, e_{12} + e_{43} + e_{22} + e_{44}, e_{11} + e_{33}, e_{14} - e_{23} + e_{32} - e_{41})$ and take note of the respective coefficients of $e_{32} + e_{41}$ and e_{42} to obtain

$$\alpha_3 + \alpha_4 - \alpha_6 + \alpha_7 = 0, \quad (32)$$

$$\alpha_2 - \alpha_3 + 4\alpha_4 - \alpha_7 + \alpha_9 = 0. \quad (33)$$

The coefficient of e_{13} in $f(e_{14} - e_{23}, e_{12} + e_{43}, e_{11} + e_{33}, I)$ yields

$$\sum_{i=1}^{10} \alpha_i = 0. \quad (34)$$

We also need the relation given by the coefficient of $e_{12} - e_{43}$ in $f(e_{11} + e_{33}, e_{11} + e_{33} + e_{12} + e_{43}, I + e_{21} + e_{34}, e_{14} - e_{23} + e_{32} - e_{41})$:

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 - 3\alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} = 0 \quad (35)$$

Subtracting (35) from (34) gives $4\alpha_6 = 0$ hence $\alpha_6 = 0$; put this in (32) to get

$$\alpha_3 + \alpha_4 + \alpha_7 = 0, \quad (36)$$

and use (30), (31) and $\alpha_6 = 0$ in (34) to obtain $\alpha_2 + \alpha_4 + \alpha_9 = 0$ which in turn, combined with (33) yields

$$\alpha_3 - 3\alpha_4 + \alpha_7 = 0; \quad (37)$$

now (36) and (37) yield $\alpha_4 = 0$, and hence $\alpha_3 = -\alpha_7$ and $\alpha_2 = -\alpha_9$. We can now rewrite (30), (31) as

$$\alpha_1 + \alpha_5 + \alpha_8 + \alpha_{10} = 0. \quad (38)$$

We get rid of four coefficients in f and write

$$\begin{aligned} f(x_1, x_2, x_3, y) &= \alpha_1 y^2 \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \right) \\ &+ \alpha_2 \left[y \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} y x_{\sigma(2)} x_{\sigma(3)} \right) - \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} y x_{\sigma(3)} \right) y \right] \\ &+ \alpha_3 \left[y \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} y x_{\sigma(3)} \right) - \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} y x_{\sigma(2)} x_{\sigma(3)} \right) y \right] \\ &+ \alpha_5 \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} y^2 x_{\sigma(2)} x_{\sigma(3)} \right) + \alpha_8 \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} y^2 x_{\sigma(3)} \right) \\ &+ \alpha_{10} \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \right) y^2. \end{aligned}$$

The coefficients of $e_{12} + e_{14}$ and e_{43} in $f(e_{14} - e_{23} + e_{12} + e_{43}, e_{32} - e_{41} + e_{21} + e_{34}, e_{11} + e_{33}, I + e_{14} - e_{23})$ produce the relations

$$4\alpha_1 - 2\alpha_2 + 6\alpha_3 - 6\alpha_5 + 2\alpha_8 = 0, \quad (39)$$

$$2\alpha_2 - 6\alpha_3 + 2\alpha_5 - 6\alpha_8 + 4\alpha_{10} = 0. \quad (40)$$

Adding (39) and (40) gives the relation $\alpha_1 - \alpha_3 - \alpha_5 - \alpha_8 + \alpha_{10} = 0$, which when subtracted from (38) results in

$$\alpha_3 + 2\alpha_5 + 2\alpha_8 = -\alpha_7 + 2\alpha_5 + 2\alpha_8 = 0. \quad (41)$$

Adding the second part of (41) to (31) gives $3\alpha_5 + 2\alpha_8 + \alpha_{10} = 0$, which combines with (38) to yield

$$\alpha_1 - 2\alpha_5 - \alpha_8 = 0. \quad (42)$$

The coefficients of e_{24} and e_{31} in $f(e_{32} - e_{41} + e_{14} - e_{23}, e_{14} - e_{23} + e_{32} - e_{41} + e_{11} + e_{33}, e_{11} + e_{33} + e_{21} + e_{34}, e_{21} + e_{34} + e_{22} + e_{44})$ yield $\alpha_1 = -\alpha_{10}$ and $\alpha_5 = -\alpha_8$ respectively. In light of this, (41) implies $\alpha_3 = \alpha_7 = 0$ and (42) implies $\alpha_1 = -\alpha_8$, thus $\alpha_1 = \alpha_5 = -\alpha_8 = -\alpha_{10}$. Using these relations, we can express f so that it has only the coefficients α_1 and α_2 . In our last substitution $f(e_{32} - e_{41} + e_{14} - e_{23}, e_{14} - e_{23} + e_{11} + e_{33}, e_{11} + e_{33} + e_{12} + e_{43}, e_{21} + e_{34} + e_{22} + e_{44})$, the coefficient of $e_{14} - e_{23} + e_{32} - e_{41}$ gives $\alpha_2 = -2\alpha_1$. Thus,

$$\begin{aligned} f(x_1, x_2, x_3, y) &= \alpha \left[y^2 \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \right) \right. \\ &\quad - \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \right) y^2 + \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} y^2 x_{\sigma(2)} x_{\sigma(3)} \right) \\ &\quad - \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} y^2 x_{\sigma(3)} \right) - 2y \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} y x_{\sigma(2)} x_{\sigma(3)} \right) \\ &\quad \left. + 2 \left(\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} y x_{\sigma(3)} \right) y \right] \\ &= \left(\frac{\alpha}{4} \right) [r_5(x_1, x_3, y, y, x_2) - r_5(x_1, x_2, y, y, x_3) + r_5(x_2, x_1, y, x_3, y) \\ &\quad - r_5(x_2, x_3, y, x_1, y) + r_5(x_3, x_2, y, x_1, y) - r_5(x_3, x_1, y, x_2, y)]. \end{aligned}$$

Thus, f is a consequence of r_5 . This completes the proof. \square

For the $*$ -identities of type 2,3, 3,2 and 4,1, we will simply state the results and briefly outline the proofs.

Theorem 3.5. *Under the hypotheses of Theorem 3.1 on the field F , any multilinear identity $f(x_1, x_2, y_1, y_2, y_3)$ of degree 5 with $x_i \in H_4(F, s)$, $y_j \in K_4(F, s)$ is a consequence of*

$$p_2(x_1, x_2, y_1, y_2, y_3) = [[x_1, y_1] \circ [x_2, y_2], y_3].$$

PROOF. As we examine homogeneous identities, the cases [5] and [4, 1] are irrelevant. On a case-by-case basis, the remaining possibilities lead to the following results:

$$\begin{aligned} [3, 2] &: - p_2(x, x, y, y, y), \\ [3, 1, 1] &: \text{no such identities,} \\ [2, 2, 1] &: p_2(x, x, y_1, y_1, y_2), \\ &\quad - p_2(x, x, y_1, y_2, y_1), \\ [2, 1, 1, 1] &: (\text{with skew-symmetric quadratic variable}) \\ &\quad p_2(x_1, x_2, y_1, y_2, y_1) - p_2(x_2, x_1, y_1, y_2, y_1), \\ [2, 1, 1, 1] &: (\text{with symmetric quadratic variable}) \\ &\quad \text{no such identities,} \\ [1, 1, 1, 1, 1] &: \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma p_2(x_1, x_2, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}). \end{aligned}$$

Thus, identities of type 2,3 are consequences of p_2 as desired. \square

Theorem 3.6. *Under the hypotheses of Theorem 3.1 on the field F , any multilinear identity $f(x_1, x_2, x_3, y_1, y_2)$ of degree 5 with $x_i \in H_4(F, s)$, $y_j \in K_4(F, s)$ is a consequence of*

$$\begin{aligned} p_2(x_1, x_2, y_1, y_2, x_3) &= [[x_1, y_1] \circ [x_2, y_2], x_3], \\ r_1(x_1, x_2, x_3, y_1, y_2) &= x_3 \mathbf{D}_{[x_1, y_2], [x_2, y_1]} \\ &\quad + \sum_{\sigma \in \mathcal{S}_2} (-1)^\sigma ([x_{\sigma(1)}, y_{\sigma(2)}], y_{\sigma(1)}) \mathbf{D}_{x_3, x_{\sigma(2)}} \\ &\quad + [x_3, y_{\sigma(1)}] \mathbf{D}_{[x_{\sigma(1)}, y_{\sigma(2)}], x_{\sigma(2)}), \\ r_2(x_1, x_2, x_3, y_1, y_2) &= \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (\{x_{\sigma(1)}, [x_{\sigma(2)}, y_1], [x_{\sigma(3)}, y_2]\}) \\ &\quad + \{x_{\sigma(1)}, [x_{\sigma(2)}, y_2], [x_{\sigma(3)}, y_1]\} + 2[x_{\sigma(1)}, y_1] \circ \langle x_{\sigma(2)}, y_2, x_{\sigma(3)} \rangle \\ &\quad + 2[x_{\sigma(1)}, y_2] \circ \langle x_{\sigma(2)}, y_1, x_{\sigma(3)} \rangle. \end{aligned}$$

PROOF. Once again, the cases [5] and [4, 1] are irrelevant. On a case-by-case basis, the remaining possibilities lead to the following results:

$$\begin{aligned}
[3, 2] &: -\left(\frac{1}{2}\right) p_2(x, x, y, y, x), \\
[3, 1, 1] &: \left(\frac{1}{2}\right) r_1(x, x, x, y_1, y_2), \\
[2, 2, 1] &: -\left(\frac{1}{2}\right) p_2(x_1, x_1, y, y, x_2), \\
&\quad p_2(x_1, x_2, y, y, x_1), \\
&\quad p_2(x_2, x_1, y, y, x_1) + \left(\frac{1}{2}\right) p_2(x_1, x_1, y, y, x_2) - r_1(x_1, x_2, x_1, y, y), \\
[2, 1, 1, 1] &: (\text{with skew-symmetric quadratic variable}) \\
&\quad -\left(\frac{1}{2}\right) r_2(x_1, x_2, x_3, y, y), \\
&\quad \left(\frac{1}{2}\right) [r_1(x_1, x_3, x_2, y, y) - r_1(x_1, x_2, x_3, y, y) \\
&\quad\quad + r_1(x_3, x_2, x_1, y, y)] - 2r_2(x_1, x_2, x_3, y, y), \\
[2, 1, 1, 1] &: (\text{with symmetric quadratic variable}) \\
&\quad -\left(\frac{1}{2}\right) [r_1(x_1, x_1, x_2, y_1, y_2) + r_1(x_2, x_1, x_1, y_1, y_2) + r_1(x_1, x_2, x_1, y_1, y_2)], \\
&\quad - [r_1(x_1, x_2, x_1, y_1, y_2) + r_1(x_2, x_1, x_1, y_1, y_2)], \\
&\quad \left(\frac{1}{2}\right) [r_1(x_1, x_1, x_2, y_1, y_2) - r_1(x_1, x_2, x_1, y_1, y_2) - r_1(x_2, x_1, x_1, y_1, y_2)] \\
&\quad\quad + p_2(x_1, x_2, y_1, y_2, x_1) - p_2(x_2, x_1, y_1, y_2, x_1), \\
[1, 1, 1, 1, 1] &: \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma p_2(x_{\sigma(1)}, x_{\sigma(2)}, y_1, y_2, x_{\sigma(3)}).
\end{aligned}$$

Thus, identities of type 3,2 are consequences of p_2 , r_1 and r_2 as desired. \square

Theorem 3.7. *Under the hypotheses of Theorem 3.1 on the field F , any multilinear identity $f(x_1, x_2, x_3, x_4, y)$ of degree 5 with $x_i \in H_4(F, s)$, $y \in K_4(F, s)$ is a consequence of*

$$\begin{aligned}
p_4(x_1, x_2, x_3, x_4, y) &= [[x_1, x_2] \circ [x_3, x_4] + [x_1, x_4] \circ [x_3, x_2], y], \\
p_6(x_1, x_2, x_3, x_4, y) &= [[x_1, x_2] \circ [x_4, y], x_3] + [[x_1, x_3] \circ [x_4, y], x_2], \\
r_3(x_1, x_2, x_3, x_4, y) &= [[y, x_3], [x_4, [x_1, x_2]]] + [[y, x_4], [x_3, [x_1, x_2]]] \\
&\quad + \sum_{\sigma \in \mathcal{S}_2} (-1)^\sigma ([x_{\sigma(1)}, [x_{\sigma(2)} \cup_{x_3, x_4}, y]] + [x_{\sigma(1)}, [y, x_{\sigma(2)}] \cup_{x_3, x_4}] \\
&\quad\quad + [\{x_3, [y, x_4], x_{\sigma(1)}\}, x_{\sigma(2)}] + [\{x_4, [y, x_3], x_{\sigma(1)}\}, x_{\sigma(2)}]), \\
r_4(x_1, x_2, x_3, x_4, y) &= \sum_{\sigma \in \mathcal{S}_4} (-1)^\sigma \left(\frac{1}{2} [x_{\sigma(1)}, [[x_{\sigma(2)}, x_{\sigma(3)}], [x_{\sigma(4)}, y]] \right. \\
&\quad\quad \left. + [\{x_{\sigma(1)}, x_{\sigma(2)}\}, y, [x_{\sigma(3)}, x_{\sigma(4)}]] \right).
\end{aligned}$$

PROOF. This time, [3, 2] is the only irrelevant case. The remaining possibilities yielded the following results:

$$\begin{aligned}
[4, 1] : & \quad \text{no such identities ,} \\
[3, 1, 1] : & \quad - \left(\frac{1}{4}\right) r_3(x_1, x_2, x_1, x_1, y), \\
[2, 2, 1] : & \quad - [p_6(x_1, x_2, x_1, x_2, y) + p_6(x_2, x_1, x_2, x_1, y)], \\
& \quad - [p_6(x_1, x_2, x_1, x_2, y) + p_6(x_2, x_2, x_1, x_1, y) + \left(\frac{1}{2}\right) p_4(x_1, x_2, x_2, x_1, y)], \\
[2, 1, 1, 1] : & \quad r_3(x_2, x_3, x_1, x_1, y) + \left(\frac{1}{2}\right) r_3(x_2, x_1, x_3, x_1, y) \\
& \quad - \left(\frac{1}{2}\right) r_3(x_3, x_1, x_1, x_2, y) - 2p_6(x_3, x_1, x_2, x_1, y) - p_6(x_1, x_2, x_3, x_1, y), \\
& \quad \left(\frac{1}{2}\right) r_3(x_2, x_3, x_1, x_1, y) + \left(\frac{1}{2}\right) r_3(x_2, x_1, x_3, x_1, y) \\
& \quad - \left(\frac{1}{2}\right) r_3(x_3, x_1, x_1, x_2, y) - 2p_6(x_3, x_1, x_2, x_1, y) - p_6(x_1, x_2, x_3, x_1, y), \\
& \quad r_3(x_2, x_3, x_1, x_1, y) - 2p_6(x_3, x_1, x_2, x_1, y) - p_6(x_1, x_2, x_3, x_1, y), \\
[1, 1, 1, 1, 1] : & \quad r_4(x_1, x_2, x_3, x_4, y).
\end{aligned}$$

Therefore, all identities of type 4,1 are consequences of p_4 , p_6 , r_3 and r_4 as desired. \square

The present paper along with [3] gives a complete picture of the minimal degree $*$ -identities of $M_n(F, *)$ for $n < 5$. In a subsequent paper, we shall tackle the case $n \geq 5$.

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