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*-Polynomial Identities of Matrices with the Transpose Involution: the Low Degrees.

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1. Introduction.

Let A be an associative algebra with involution $*$ over a field F . Amitsur [2] defined a $*$-polynomial to be a polynomial where the variables can also appear with a $*$; such polynomials can be evaluated on associative algebras with involution in the obvious way. When $A = M_n(F)$, the $n \times n$ matrices with entries in F, the ideal $I(M_n(F), *)$ of $*$-identities coincides with either $I(M_n(F), t)$ for t the transpose involution or $I(M_n(F), s)$ for s the symplectic involution (n even) [16, Thm. 3.1.62]. In both cases, the question of what is the minimal degree of a $*$-polynomial identity is still open. Giambruno [4, Thm. 1] has shown that

Theorem 1.1. *Over a field F of characteristic not 2, if f is a $*$-polynomial identity for $M_n(F)$ and $n > 2$, then the degree of f is greater than n.* □

While in general this result is not sharp, as we will see, it is for $n = 3$ or 4 with $*$ the transpose involution. The $*$-identities for $M_2(F)$, F of characteristic 0, have been completely determined by Levchenko, Drensky and Giambruno [3, 7, 8] for both types of involutions. Our purpose is to determine the $*$-identities of minimal degree for $(M_n(F), t)$ when $n < 5$. We have reason to believe that the situation for low n's is atypical and that it must be studied carefully before the general case can be dealt with.

There are two aspects to our results: producing the identities and showing that there are no others. The existence results hold for arbitrary fields and are proved using the fact that matrix algebras or some subspaces are in some sense of degree 1 or 2. For the uniqueness results we must impose

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some mild characteristic restrictions and these proofs ultimately depend on substitutions.
Our starting point is, as always, the *standard polynomial*

$$\mathrm{S}_n(x_1, x_2, \ldots, x_n) := \sum_{\sigma \in \mathcal{S}_n} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}, \tag{1.2}$$

where the sum is taken over the *symmetric group* $\mathcal{S}_n$. Amitsur and Levitzki [1] have shown that $M_n(F)$ has no polynomial identity of degree less than 2n, that S_{2n} is an identity of $M_n(F)$ and that if $|F| > 2$ or $n > 2$, then any identity of $M_n(F)$ of degree $2n$ is a scalar multiple of S_{2n}. Let $H_n(F)$ denote the symmetric matrices and $K_n(F) = \{a - a^t \mid a \in M_n(F)\}$. If the characteristic of F is not 2 then $K_n(F)$ is the set of skew-symmetric matrices and when the characteristic is 2, it coincides with the set of symmetric matrices with 0's along the diagonal. Since we are working over a fixed but arbitrary field, we will often use M_n, H_n and K_n.

If the characteristic is not 2, the existence of a $*$-identity of a given degree d in an algebra A is equivalent to the existence of a polynomial of the same degree which yields 0 for every substitution where the last k variables are replaced by elements of $H(A, *)$ and the remaining $d - k$ variables are replaced by elements of $K(A, *)$, for k some integer between 0 and d (see for example [4]). Accordingly, we make the following

Definition 1.3. An (n, d, s)-*identity* is a multilinear $*$-polynomial identity of M_n

$$f(z_1, \ldots, z_d) = \sum_{\sigma \in \mathcal{S}_d} \alpha_\sigma z_{\sigma(1)} \ldots z_{\sigma(d)},$$

where $z_i = x_i - x_i^*$ for $1 \le i \le d - s$, and $z_i = x_i + x_i^*$ for $d - s + 1 \le i \le d$.

Hence an (n, d, s)-*identity* is a multilinear identity of M_n of degree d with s "symmetric" variables and $d - s$ "skew-symmetric " variables. Let

$$\{a\ b\ c\} := abc + cba, \qquad \text{for} \quad a, b, c \in A. \tag{1.4}$$

This triple product induces a Jordan triple system structure on A, $H(A, *)$, and $K(A, *)$ [11]. Next, let

$$[a,\ b] := ab - ba, \qquad a \circ b := ab + ba, \qquad a\mathrm{U}_b := bab. \tag{1.5}$$

The Lie bracket induces a Lie algebra structure on A and $K(A, *)$ and the circle product induces a Jordan algebra structure on A and $H(A, *)$.

For a unital commutative ring R, the following identities are mentioned by Rowen in [16, Remark 2.5.15]:

$$\begin{aligned} x_1 - x_1^* &\in I(M_1(R), t); \\ [x_1 - x_1^*,\ x_2 - x_2^*] &\in I(M_2(R), t); \\ [\mathrm{S}_3(x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*),\ x_4] &\in I(M_3(R), t); \\ \mathrm{S}_6(x_1 - x_1^*, \ldots, x_6 - x_6^*) &\in I(M_4(R), t); \\ [x_1 + x_1^*,\ x_2] &\in I(M_2(R), s); \\ [[x_1 + x_1^*,\ x_2 - x_2^*]^2,\ x_3] &\in I(M_4(R), s). \end{aligned}$$

Then Rowen cautiously adds: "As far as I know, all minimal identities of $(M_n(\mathbf{Z}), *)$ (for $n \leq 4$) are implied by these identities." As we will see below, more polynomials are required to obtain all the identities of minimal degree, at least in the case of the transpose involution.

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2. *-Identities of Minimal Degree for $M_n(F)$, $n < 5$.

In this section we exhibit some identities for $K_n(F)$, $n < 5$, and some $*$-identities for $M_3(F)$ involving one symmetric variable; in the next section we show that these suffice to obtain all minimal degree $*$-identities under some mild characteristic restrictions.

We recall that Kostant [5, 6] has shown that, for n even, S_{2n-2} is an identity of $K_n(F)$ but S_{2n-3} isn't. This was extended to all n's by Rowen [15, 17]. Note that while this result says that S_{2n-2} is the minimal degree standard identity satisfied by $K_n(F)$, it does not say that $K_n(F)$ satisfies no other identity of degree $2n-2$ or less. This also implies that $\mathrm{S}_{2n-2}(x_1 - x_1^*, \ldots, x_{2n-2} - x_{2n-2}^*)$ is a $*$-polynomial identity of $M_n(F)$. Since the minimal degree of an identity of $H_n(F)$ is $2n$ [10] and the minimal degree of an identity of $K_n(F)$ is less than or equal to $2n-2$ [5, 6, 15, 17], the polynomial corresponding to $*$-identities of minimal degree must involve some "skew-symmetric variables". In view of some of Rowen's results, we expect most if not all of them to be skew.

We begin with $n = 1$ in which case it is trivial to see that $(x_1 - x_1^*) \in I(M_1)$. If $n = 2$, $K_2(F)$ has dimension 1 and so

$$\mathrm{S}_2 \in I(K_2) \text{ and hence } [x_1 - x_1^*,\ x_2 - x_2^*] \in I(M_2). \tag{2.1}$$

We now turn to $n = 3$. Let

$$r(x_1, x_2, x_3, x_4) := [\mathrm{S}_3(x_1, x_2, x_3),\ x_4]. \tag{2.2}$$

Since S_3 is multilinear alternating of degree 3 and $K_3(F)$ has dimension 3, it is easy to check that S_3 is a central polynomial on K_3 (in fact it is an identity on K_3 in characteristic 2) and so $r(x_1, x_2, x_3, x_4) \in I(K_3)$. The dimensionality argument also shows that $\mathrm{S}_4 \in I(K_3)$, a fact already known from the Kostant-Rowen results. Next, we point out that one could choose to view $K_3(F)$ as a Jordan triple system which happens to be isomorphic to 1×3 matrices over F and is therefore of degree 1. But more directly, an easy computation yields

$$\begin{aligned} \mathrm{ABA} = -(a_{12}b_{12}+&a_{13}b_{13} + a_{23}b_{23})\mathrm{A}, \\ &\text{for} \quad \mathrm{A} = (a_{ij}), \quad \mathrm{B} = (b_{ij}) \in K_3(F). \end{aligned} \tag{2.3}$$

Irrespective of one's point of view, (2.3) yields

$$[\mathrm{ABA},\ \mathrm{A}] = 0, \qquad \text{for} \quad \mathrm{A},\ \mathrm{B} \in K_3(F),$$

and $[xyx,\ x]$ is an identity of K_3. To describe our next identity, we use tr to denote the *trace* and we linearize (2.3) to get

$$\{\mathrm{A\ B\ C}\} = \frac{1}{2}\mathrm{tr}(\mathrm{AB})\mathrm{C} + \frac{1}{2}\mathrm{tr}(\mathrm{CB})\mathrm{A}, \quad \text{for} \quad \mathrm{A},\ \mathrm{B},\ \mathrm{C}\ \in K_3(F), \tag{2.4}$$

where the half is purely formal and makes sense even in characteristic 2 (tr(AB) is a linear combination of terms whose coefficients are even integers). Now we consider

$$p(x_1, x_2, x_3, x_4) := \sum_{(123)} \{x_1\ [x_2,\ x_4]\ x_3\}, \tag{2.5}$$

where $\sum_{(123)}$ denotes the sum over the cyclic permutations of 1, 2, 3. If we assume that (2.4) holds formally, then $p(x_1, x_2, x_3, x_4)$ is a linear combination of x_1, x_2 and x_3 in which the coefficient of x_3 is

$$\frac{1}{2}(\mathrm{tr}(x_1[x_2,\ x_4]) + \mathrm{tr}(x_2[x_1,\ x_4])) = \frac{1}{2}\mathrm{tr}([x_1x_2,\ x_4]) = 0.$$

Since $p(x_1, x_2, x_3, x_4)$ is symmetric in x_1, x_2 and x_3, $p(x_1, x_2, x_3, x_4) \in I(K_3)$. Our last key polynomial for K_3 is

$$
\begin{aligned}
q(x_1,x_2,x_3,x_4) := \sum_{(123)}\{x_1\ [x_2,\ x_3]\ x_4\} &+ \sum_{(124)}\{x_1\ [x_2,\ x_4]\ x_3\} \\
&+ 2([x_1,\ x_3]\mathrm{D}_{x_2,x_4} + [x_1,\ x_4]\mathrm{D}_{x_2,x_3} \\
&- [x_2,\ x_3]\mathrm{D}_{x_1,x_4} - [x_2,\ x_4]\mathrm{D}_{x_1,x_3}),
\end{aligned} \tag{2.6}
$$

where D is the derivation

$$x\mathrm{D}_{y,z} := \{x\ y\ z\} - \{x\ z\ y\}.$$

The polynomial q is alternating in x_1, x_2 and symmetric in x_3, x_4. We claim that q is an identity on K_3. Again assuming that (2.4) holds formally, we note that

$$
\begin{aligned}
2[x_1,\ x_3]\mathrm{D}_{x_2,x_4} &= \mathrm{tr}([x_1,\ x_3]x_2)x_4 + \mathrm{tr}(x_4x_2)[x_1,\ x_3] \\
&\qquad - \mathrm{tr}([x_1,\ x_3]x_4)x_2 - \mathrm{tr}(x_2x_4)[x_1,\ x_3] \\
&= \mathrm{tr}([x_1,\ x_3]x_2)x_4 - \mathrm{tr}([x_1,\ x_3]x_4)x_2.
\end{aligned}
$$

Therefore $q(x_1,x_2,x_3,x_4)$ is a linear combination of x_i, $1 \le i \le 4$, and by symmetry we need only consider the coefficients of x_2 and x_4. The coefficient of x_2 is

$$
\begin{aligned}
&\frac{1}{2}(\mathrm{tr}(x_4[x_3,\ x_1]) + \mathrm{tr}(x_3[x_4,\ x_1])) - \mathrm{tr}([x_1,\ x_3]x_4) - \mathrm{tr}([x_1,\ x_4]x_3) \\
&\qquad = \frac{1}{2}\mathrm{tr}([x_3x_4,\ x_1]) - \mathrm{tr}([x_1,\ x_3x_4]) = 0.
\end{aligned}
$$

The coefficient of x_4 is

$$
\begin{aligned}
&\frac{1}{2}(\mathrm{tr}(x_1[x_2,\ x_3]) + \mathrm{tr}(x_2[x_3,\ x_1]) + \mathrm{tr}(x_3[x_1,\ x_2]) + \mathrm{tr}([x_1,\ x_2]x_3)) \\
&\qquad + \mathrm{tr}([x_1,\ x_3]x_2) - \mathrm{tr}([x_2,\ x_3]x_1);
\end{aligned}
$$

collecting terms, this equals

$$
\begin{aligned}
&\frac{1}{2}(\mathrm{tr}(x_1[x_2,\ x_3]) + \mathrm{tr}(x_2[x_1,\ x_3])) + \mathrm{tr}([x_1,\ x_2]x_3) - \mathrm{tr}([x_2,\ x_3]x_1) \\
&\qquad = \frac{1}{2}\mathrm{tr}([x_1x_2,\ x_3]) + \mathrm{tr}([x_1x_3,\ x_2]) = 0.
\end{aligned}
$$

Hence $q(x_1,x_2,x_3,x_4) \in I(K_3)$ and we have proved

Proposition 2.7. *The polynomials*

$$\begin{aligned} p(x_1,x_2,x_3,x_4) &= \sum_{(123)} \{x_1\ [x_2,\ x_4]\ x_3\}, \\ q(x_1,x_2,x_3,x_4) &= \sum_{(123)} \{x_1\ [x_2,\ x_3]\ x_4\} + \sum_{(124)} \{x_1\ [x_2,\ x_4]\ x_3\} \\ &\qquad + 2([x_1,\ x_3]\mathrm{D}_{x_2,x_4} + [x_1,\ x_4]\mathrm{D}_{x_2,x_3} \\ &\qquad - [x_2,\ x_3]\mathrm{D}_{x_1,x_4} - [x_2,\ x_4]\mathrm{D}_{x_1,x_3}), \\ r(x_1,x_2,x_3,x_4) &= [\mathrm{S}_3(x_1,x_2,x_3),\ x_4] \end{aligned}$$

are identities of $K_3(F,t)$. □

There are a few more symmetry considerations worth noting about p, q and r. For example, the polynomial r is alternating in the first 3 variables. Hence, up to a sign, the only homogeneous polynomial obtainable from r by allowing some of the variables to be equal is

$$\begin{aligned} r(x_1,x_2,y,y) = -y^2[x_1,\ x_2] + y < x_1\ y\ x_2 > \\ - < x_1\ y\ x_2 > y + [x_1,\ x_2]y^2, \end{aligned} \tag{2.8}$$

in which we use the notation

$$< a\ b\ c > := abc - cba, \qquad \text{for} \quad a,b,c \in A. \tag{2.9}$$

We also have

$$\begin{aligned} r(x_1,x_2,x_3,x_4) - r(x_1,x_2,x_4,x_3) - r(x_1,x_4,x_3,x_2) - r(x_4,x_2,x_3,x_1) \\ = 2\mathrm{S}_4(x_1,x_2,x_3,x_4). \end{aligned} \tag{2.10}$$

We already noted that the polynomial q is alternating in x_1 and x_2 and symmetric in x_3 and x_4. One can also check that $q(x_1,y,y,y) = 0$ while

$$\begin{aligned} q(x_1,x_2,y,y) = 2y^2[x_1,\ x_2] + 6y < x_1\ y\ x_2 > -4y[x_1,\ x_2]y \\ + 6 < x_1\ y\ x_2 > y + 2[x_1,\ x_2]y^2 - 8 < x_1\ y^2\ x_2 > . \end{aligned} \tag{2.11}$$

Finally, by setting some of the variables equal in $p(x_1, x_2, x_3, x_4)$, we obtain

$$p(y, y, x, y) = \{y\ [x,\ y]\ y\} = [\{y\ x\ y\},\ y] = 2[yxy,\ y], \tag{2.12}$$

and

$$\begin{aligned} p(y, y, x_1, x_2) - p(y, y, x_2, x_1) = &-2y^2[x_1,\ x_2] + 2y < x_1\ y\ x_2 > \\ &+ 4y[x_1,\ x_2]y + 2 < x_1\ y\ x_2 > y - 2[x_1,\ x_2]y^2. \end{aligned} \tag{2.13}$$

Remarks 2.14. 1. The polynomials p, q and r generate an $F[\mathcal{S}_4]$-module under the action which permutes the variables. One can check that if the characteristic of F is not 2 or 3 then the character of this representation is [3,1] + 2[2,1,1] + [1,1,1,1] in the usual notation for Young diagrams.

2. In a certain sense, $[xyx,\ x] \in I(K_3)$ was already known. In [14, Thm. 2] it is shown that $(\mathrm{ad}_x)^3$ acts as a derivation on the Lie algebra sl_2 which is isomorphic to K_3. Expressing this as a polynomial identity of the adjoint representation yields $[\mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x,\ \mathrm{ad}_x]$.

Consider now the polynomial

$$g(x_1, x_2, x_3, x_4) = \sum_{\sigma \in \mathcal{S}_2} (-1)^\sigma (\{x_{\sigma(1)}\ x_{\sigma(2)}\ (x_3 \circ x_4)\} - \{x_{\sigma(1)}\ (\ x_{\sigma(2)} \circ x_4)\ x_3\}) \tag{2.15}$$

in which x_1, x_2, x_3 are skew-symmetric and x_4 is symmetric. Using $K_n \circ H_n \subset K_n$ and equation (2.4), one can verify that

$$\begin{aligned} g(x_1, x_2, x_3, x_4) = &\frac{1}{2}((tr((x_3 \circ x_4)x_2) - tr(x_3(x_2 \circ x_4)))x_1 \\ &+ (tr(x_3(x_1 \circ x_4)) - tr((x_3 \circ x_4)x_1))x_2 \\ &+ (tr(x_2(x_1 \circ x_4)) - tr(x_1(x_2 \circ x_4)))x_3 \\ &+ (tr(x_1 x_2) - tr(x_2 x_1))(x_3 \circ x_4)) \\ = &\ 0. \end{aligned}$$

As was already mentioned, $\mathrm{S}_3(x_1, x_2, x_3)$ (for $x_i \in K_3$) is central in M_3 and therefore we have

Proposition 2.16. *The polynomials*

$$g(x_1,x_2,x_3,x_4) = \sum_{\sigma\in\mathcal{S}_2} (-1)^\sigma (\{x_{\sigma(1)}\ x_{\sigma(2)}\ (x_3\circ x_4)\} - \{x_{\sigma(1)}\ (\ x_{\sigma(2)}\circ x_4)\ x_3\}),$$
$$r(x_1,x_2,x_3,x_4) = [\mathrm{S}_3(x_1,x_2,x_3),\ x_4]$$

are $(3,4,1)$*-identities.* □

Remark 2.17. Rowen [15] has shown that for n odd, S_{2n-2} is an $(n,2n-2,1)$-identity. An elementary proof for $n=3$ is given by the relation

$$\mathrm{S}_4(x_1,x_2,x_3,x_4) = 2r(x_1,x_2,x_3,x_4) - h(x_1,x_2,x_3,x_4),$$

where

$$h(x_1,x_2,x_3,x_4) = \sum_{(123)} (x_4\circ x_3)D_{x_2,x_1}.$$

To see that h is an identity, it suffices, by symmetry considerations, to show that the coefficients of $x_3\circ x_4$ and x_3 vanish; the former is $\frac{1}{2}(tr(x_1x_2) - tr(x_2x_1)) = 0$ and the latter is $\frac{1}{2}(tr((x_2\circ x_4)x_1) - tr((x_1\circ x_4)x_2)) = 0$.

We now handle $n=4$. We will later see that in this case, there are no mixed identities and so we only need to examine K_4. In order to present K_4's identities in their proper context, we first recall a general construction of a "degree 2" Jordan triple system (see [11]). Let L be an extension of F with involution $^-$, that is an F-linear automorphism of period 2, V an L-module, $Q: V\to L$ a quadratic form and $s: V\to V$ an L-linear map satisfying

$$s^2 = \mathrm{Id},\quad s(cv) = \bar{c}s(v) \text{ and } Q(sv) = \overline{Q(v)},\quad \text{for all } c\in L,\ v\in V. \tag{2.18}$$

Then

$$\{u\ v\ w\} := Q(u,\ sv)w + Q(w,\ sv)u - Q(u,\ w)sv, \tag{2.19}$$

(where $Q(u,\ v)$ is the bilinearization of Q) defines a Jordan triple product on V. Such Jordan triple systems are referred to as *quadratic form triples* [11, Example 1.6]. By (2.18),

$$Q(su,\ v) = \overline{Q(ssu,\ sv)} = \overline{Q(u,\ sv)}. \tag{2.20}$$

In particular let $L = F$ (so $^-$ is the identity map) and $V = K_4(F)$. The Pfaffian [11] is a nondegenerate quadratic form on K_4; it is a square root of the determinant and so is determined up to sign. We make the following choice:

$$\mathrm{Pf\ A} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}, \qquad \text{for A} = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix}.$$

For A as above, define the following involutory map on K_4,

$$s\mathrm{A} = \begin{bmatrix} 0 & -a_{34} & a_{24} & -a_{23} \\ a_{34} & 0 & -a_{14} & a_{13} \\ -a_{24} & a_{14} & 0 & -a_{12} \\ a_{23} & -a_{13} & a_{12} & 0 \end{bmatrix}.$$

One checks that

$$\mathrm{ABA} = \mathrm{Pf(A,\ sB)A} - \mathrm{Pf(A)\ sB},$$

where Pf(A, B) is the bilinearization of the Pfaffian. Therefore K_4 with the usual triple product is a quadratic form triple. The Jordan nature of K_4 prompts us to revisit the polynomial

$$\kappa(x_1, x_2, x_3, y) := y^2\mathrm{S}_3(\mathrm{V}_{x_1}, \mathrm{V}_{x_2}, \mathrm{V}_{x_3}) - (y\mathrm{S}_3(\mathrm{V}_{x_1}, \mathrm{V}_{x_2}, \mathrm{V}_{x_3})) \circ y \tag{2.21}$$

(where $y\mathrm{V}_x := y \circ x$), which has been shown (by the second author in [13]) to be an identity for all Jordan algebras of degree 2. The expression in (2.21) can be rewritten in associative terms

$$\kappa(x_1, x_2, x_3, y) = \mathrm{S}_4(x_1, x_2, x_3, y^2) - y \circ \mathrm{S}_4(x_1, x_2, x_3, y). \tag{2.22}$$

or in terms of linear triple products

$$\begin{aligned} \kappa(x_1, x_2, x_3, y) := \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (\{\{y\ y\ x_{\sigma(1)}\}\ x_{\sigma(2)}\ x_{\sigma(3)}\} \\ - \{\{y\ x_{\sigma(1)}\ x_{\sigma(2)}\}\ x_{\sigma(3)}\ y\} \\ - \{x_{\sigma(1)}\ \{y\ x_{\sigma(2)}\ x_{\sigma(3)}\}\ y\}), \end{aligned} \tag{2.23}$$

and we denote its linearization by

$$\kappa(x_1, x_2, x_3; x; y) := (x \circ y)\mathrm{S}_3(\mathrm{V}_{x_1}, \mathrm{V}_{x_2}, \mathrm{V}_{x_3}) - (x\mathrm{S}_3(\mathrm{V}_{x_1}, \mathrm{V}_{x_2}, \mathrm{V}_{x_3})) \circ y$$
$$(y\mathrm{S}_3(\mathrm{V}_{x_1}, \mathrm{V}_{x_2}, \mathrm{V}_{x_3})) \circ x. \tag{2.24}$$

The key reason why κ is an identity for degree 2 Jordan algebras is that an alternating sum vanishes whenever two of the alternating variables are also symmetric. On the other hand, if κ is evaluated on a quadratic form triple system, then the result is a linear combination of y, sy, $x_{\sigma(i)}$ and $sx_{\sigma(i)}$, $1 \le i \le 3$. Starting with (2.23) and using (2.18) and (2.19), the coefficients of y in the first, second and third terms of κ are respectively given by

$$\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma Q(x_{\sigma(1)},\ sy)Q(x_{\sigma(3)},\ sx_{\sigma(2)}), \tag{2.25}$$

$$-\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (Q(x_{\sigma(2)},\ sx_{\sigma(3)})Q(y,\ sx_{\sigma(1)}) + 2Q(y,\ sx_{\sigma(3)})Q(x_{\sigma(2)},\ sx_{\sigma(1)})$$
$$- Q(sx_{\sigma(1)},\ sx_{\sigma(3)})Q(y,\ x_{\sigma(2)})), \tag{2.26}$$

and

$$-\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (Q(x_{\sigma(1)},\ sy)\overline{Q(x_{\sigma(3)},\ sx_{\sigma(2)})} + Q(x_{\sigma(1)},\ sx_{\sigma(3)})\overline{Q(y,\ sx_{\sigma(2)})}$$
$$- Q(x_{\sigma(1)},\ x_{\sigma(2)})\overline{Q(y,\ x_{\sigma(3)})}). \tag{2.27}$$

The third summands of (2.26) and (2.27) are alternating sums of symmetric functions and therefore 0. By (2.20),

$$Q(x_{\sigma(1)},\ sy)\overline{Q(x_{\sigma(3)},\ sx_{\sigma(2)})} = Q(sx_{\sigma(3)},\ x_{\sigma(2)})\overline{Q(sx_{\sigma(1)},\ y)},$$

and so the remaining two terms of (2.27) differ only by a transposition and their alternating sum cancels. Similarly the first term of (2.26) cancels with half the second term and the coefficient of y reduces to

$$\sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma (Q(x_{\sigma(1)},\ sy)Q(x_{\sigma(3)},\ sx_{\sigma(2)}) - Q(y,\ sx_{\sigma(3)})Q(x_{\sigma(2)},\ sx_{\sigma(1)}))$$
$$= \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma Q(x_{\sigma(2)},\ sx_{\sigma(1)})(Q(sy,\ x_{\sigma(3)}) - \overline{Q(sy,\ x_{\sigma(3)})}) \tag{2.28}$$

Similarly the coefficient of sy is given by

$$\sum_{\sigma\in\mathcal{S}_3}(-1)^\sigma Q(y,\ x_{\sigma(1)})(Q(sx_{\sigma(3)},\ x_{\sigma(2)})-\overline{Q(sx_{\sigma(3)},\ x_{\sigma(2)})}). \tag{2.29}$$

As the following example will show, the expressions (2.28) and (2.29) are in general not identically 0.

Example 2.30. Let $L=F[\theta]$, a quadratic extension of a field F of characteristic not 2 with canonical involution $\overline{\theta}=-\theta$, V the direct sum of three hyperbolic planes spanned by hyperbolic pairs $u_i,\ v_i$, $1\le i\le 3$ and $s:V\to V$ the L-semilinear map given by $su_1=u_1,\ sv_1=v_1,\ su_2=u_3,\ sv_2=v_3,$ $su_3=u_2$ and $sv_3=v_2$. For $x_1=u_1,\ x_2=\theta v_1,\ x_3=u_3$ and $y=\theta v_2$, by (2.28), the coefficient of v_2 in $\kappa(x_1,x_2,x_3,y)$ is

$$\begin{aligned}(Q(u_1,\ \theta v_1)(Q(u_3,\ -\theta v_3)-\overline{Q(u_3,\ -\theta v_3)})\\ -Q(-\theta v_1,\ u_1)(Q(u_3,\ -\theta v_3)-\overline{Q(u_3,\ -\theta v_3)}))\theta\\ =-4\theta^3.\end{aligned}$$

Therefore κ is not an identity for arbitrary Jordan triple systems of a quadratic form. However we will prove

Theorem 2.31. *If $(V,Q,^-)$ is a Jordan triple system of a quadratic form Q and $^-$ is the identity map, then $\kappa(x_1,x_2,x_3,y)$ is a polynomial identity of V.*

Proof. If $^-$ is the identity map then (2.20) becomes

$$Q(su,\ v)=Q(u,\ sv)$$

and $Q(su,\ v)$ is symmetric in u and v. Therefore the alternating sums in (2.28) and (2.29) vanish. Since κ is alternating in $x_1,\ x_2,\ x_3$, it is enough to show that the coefficients of x_3 and sx_3 both vanish. Arguing as above, the coefficient of x_3 in $\kappa(x_1,x_2,x_3,y)$ is

$$3\sum_{\sigma\in\mathcal{S}_2}(-1)^\sigma Q(x_{\sigma(1)},\ sx_{\sigma(2)})Q(y,\ sy),$$

which is 0, while the coefficient of sx_3 is

$$\sum_{\sigma\in\mathcal{S}_2}(-1)^\sigma(Q(x_{\sigma(2)},\ y)(Q(x_{\sigma(1)},\ sy)-\overline{Q(x_{\sigma(1)},\ sy)})$$
$$+\,Q(sx_{\sigma(1)},\ x_{\sigma(2)})Q(y,\ y)),$$

which is also 0. □

Corollary 2.32. *$\kappa(x_1,x_2,x_3,y)$ is a polynomial identity of $K_4(F,t)$.* □

Remark 2.33. The polynomials

$$\kappa_{ij}:=\kappa(x_1,\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,x_5;x_i;x_j),\quad 1\le i<j\le 5 \tag{2.34}$$

generate $I(K_4)$ as $F[\mathcal{S}_5]$-module and the dimension is 10. One can check that if the characteristic of F is not 2 or 3, the character of this representation is [3,1,1] + [2,1,1,1] in the usual notation for Young diagrams.

æ 3. Uniqueness.

We now proceed to show that under some restrictions on the base field, any $*$-polynomial identity of M_n of minimal degree for $n<5$ can be obtained from the identities presented in the previous section. We recall that whenever the degree of each variable x_i in an identity $p(x_1,\ldots,x_m)$ is less than $|F|$, then each homogeneous component of p is also an identity. Accordingly, we will carry out a case by case study of homogeneous $*$-PI's. In order to simplify our calculations, we first introduce a useful result of Ma.

In [9], Ma defines a partial ordering on the homogeneous elements of the free associative algebra $F[X]$. A polynomial $p(x_1,\ldots,x_m)$ is said to be of type $[n_1,\ldots,n_m]$ if the n_j's are the degrees of the x_i's rearranged in nonincreasing order. (If an integer is repeated k times, we denote this with an exponent; for example, $[3,2^2,1^3]$ means $[3,2,2,1,1,1]$.) Given homogeneous polynomials p and p' of degree n and n' and of type $[n_1,\ldots,n_m]$ and $[n'_1,\ldots,n'_m]$ respectively, we say that p is lower than p' in the partial ordering if and only if either (i) $n<n'$, or (ii) $n=n'$ and $n_j>n'_j$ for the first j such that $n_j\ne n'_j$. Otherwise two polynomials are not comparable.

Now let $f(x_1,\ldots,x_r,y_1,\ldots,y_s,\ldots,z_1,\ldots,z_t)$ be a homogeneous polynomial identity of type $[m^r,n^s,\ldots,u^t]$ on some subspace V of $M_n(F)$ (e.g. $V=H_n$ or K_n) and set $m_0=\max\{m,n,\ldots,u,r,s,\ldots,t\}$. The following theorem provides a relation of symmetry between variables of equal degree in f, depending on how many there are.

Theorem 3.1. *If the characteristic of F does not divide $m_0!$ and $|F| > 2m_0 - 1$, then $f = f_0 + f_1$ where f_i, $i = 0$, 1, are identities of the same type as f and for each k with $0 < k \leq m_0$, f_0 is symmetric or skew-symmetric in all variables of degree k depending on whether k is even or odd while f_1 comes from identities of lower type.* $\square$

We can outline Ma's proof as follows. One first establishes the above result for a pair of variables x, y of some given degree m in f; then, by acting on all r variables of that same degree m with the symmetric group $\mathcal{S}_r$, one obtains the desired symmetry property among those particular r variables. Repeating the procedure for each degree separately yields the general result.

It becomes apparent that we can extend this result to (n, d, s)-identities by fixing the degree, keeping the s symmetric and $d - s$ skew-symmetric variables apart and acting on them via $\mathcal{S}_s \times \mathcal{S}_{d-s}$. Thus, when considering a typical homogeneous (n, d, s)-identity, we may assume the symmetry properties of Theorem 3.1, and with this, reduce the number of arbitrary coefficients involved in our calculations.

The case $n = 1$ being trivial ($K_n = \{0\}$ so $x - x^*$ is a $*$-identity), we begin with $n = 2$. If at least one of the variables (say x_2) is symmetric in $f(x_1, x_2) = \alpha x_1 x_2 + \beta x_2 x_1$, then $f(e_{12} \pm e_{21}, e_{22}) = \alpha e_{12} \pm \beta e_{21} = 0$ implies $\alpha = \beta = 0$; hence, the $*$-identities of M_2 of minimal degree do not involve any symmetric variables. Now $f(e_{12} - e_{21}, e_{12} - e_{21}) = 0$ implies $\alpha = -\beta$ and this with (2.1) proves

Theorem 3.2. *If $|F| > 2$, then any polynomial identity of $K_2(F, t)$ of minimal degree is a scalar multiple of $\mathrm{S}_2(x_1, x_2)$ and any $*$-polynomial identity of $(M_2(F), t)$ of minimal degree is a consequence of $\mathrm{S}_2(x_1 - x_1^*, x_2 - x_2^*)$.* $\square$

Remark 3.3. Although not of minimal degree, it is not hard to check that the polynomials $[x_1,\ x_2]x_3$, $x_3[x_1,\ x_2]$, $x_1x_2x_3 - x_3x_2x_1$ and $x_1x_3x_2 - x_2x_3x_1$ form a basis for the space of $(2, 3, 1)$-identities.

For $n = 3$ and 4, the minimal degree of a $*$-identity is $n + 1$. In these cases, the next lemma provides an upper bound on the possible number of symmetric variables involved.

Lemma 3.4. *For $n \geq 2$ and $* = t$, an $(n, n+1, s)$-identity must have $s \leq 1$.*

Proof. Suppose $f(z_1, \ldots, z_{n+1}) = \sum_{\sigma \in \mathcal{S}_{n+1}} \alpha_\sigma z_{\sigma(1)} \ldots z_{\sigma(n+1)} \neq 0$ is an $(n, n + 1, s)$-identity. We first show that $s \leq 2$, and if $s = 2$, then the two symmetric variables are adjacent in each and every monomial of f. For, suppose some monomial of f has two non-adjacent symmetric variables (e.g. when $s \geq 3$); renaming the variables and scaling if necessary, we may assume this monomial to be $z_1 z_2 \ldots z_{n+1}$ (with $\alpha_1 = 1$) where say z_i and z_j are symmetric, $j > i+1$. Set

$$\begin{aligned}
&A_k = e_{kk+1} \pm e_{k+1k}, \quad 1 \leq k \leq i-1, \qquad A_i = e_{ii}, \\
&A_k = e_{k-1k} \pm e_{kk-1}, \quad i+1 \leq k \leq j-1, \quad A_j = e_{j-1j-1}, \\
&A_k = e_{k-2k-1} \pm e_{k-1k-2}, \quad j+1 \leq k \leq n+1,
\end{aligned}$$

where we take $+$ or $-$ in A_k according as $z = x_i + x_i^*$ or $z = x_i - x_i^*$. Then

$$f(A_1, \ldots, A_{n+1}) = e_{1n} + \sum_{\substack{m \neq 1 \\ r \neq n}} \alpha_{mr} e_{mr} \neq 0,$$

a contradiction.

Next we point out that if $s = 2$ then f cannot be alternating in its symmetric variables z_n, z_{n+1}, for otherwise setting $y_i = z_i$, $1 \leq i \leq n-1$, $y_n = [z_n,\ z_{n+1}]$ we obtain $f(y_1, \ldots, y_n) \in I(K_n)$ (using $K_n = \mathrm{Span}([H_n,\ H_n])$) which contradicts Giambruno's Theorem (1.1). Hence, $p(z_1, \ldots, z_{n-1}, y) := f(z_1, \ldots, z_{n-1}, y, y) \neq 0$ is a $*$-polynomial identity of M_n whose monomials are of the form

$$z_{\sigma(1)} \ldots z_{\sigma(i-1)} y^2 z_{\sigma(i)} \ldots z_{\sigma(n-1)},$$

where y is symmetric and z_j are skew-symmetric. Assuming that $\alpha_1 = 1$ and that y occurs in the i^{th} position, set $A_j = e_{jj+1} - e_{j+1j}$, $B = e_{ii}$; then $p(A_1, \ldots, A_{n-1}, B) = e_{1n} + \ldots \neq 0$ yielding again a contradiction. Thus $s \leq 1$. □

Consider now the case $n = 3$. By (1.1), there are no $(3, d, s)$-identities for $d < 4$, and by lemma 3.4, $(3, 4, s)$-identities must have $s \leq 1$. Suppose first that $s = 0$, that is, we look at the identities of K_3. According to (3.1), we need only consider homogeneous polynomials of degree [4], [3,1], [2,2], [2,1,1] and [1,1,1,1] with the third polynomial symmetric in its 2 variables, the fourth alternating in its linear variables, and the last alternating in all its variables. The [1,1,1,1] polynomials are multiples of S_4, and so by (2.10) they are a consequence of r. Since $x^4 \notin I(K_3)$ we are left with the middle three types.

An arbitrary polynomial of type [3,1] has the form

$$f(x,y) = \alpha_1 xy^3 + \alpha_2 yxy^2 + \alpha_3 y^2 xy + \alpha_4 y^3 x.$$

Assume that $f \in I(K_3)$. Letting $x = e_{12} - e_{21}$ and $y = e_{23} - e_{32}$ yields $\alpha_1 = 0 = \alpha_4$. Letting $y = x$, using the fact that $x^4 \notin I(K_3)$ yields $\alpha_3 = -\alpha_2$. Therefore

$$f(x,y) = \alpha[yxy,\ y]$$

which, by (2.12), is a consequence of p.

Next, a polynomial of type [2,2] symmetric in its 2 variables has the form

$$f(x,y) = \alpha_1(x^2y^2 + y^2x^2) + \alpha_2(xyxy + yxyx) + \alpha_3(xy^2x + yx^2y).$$

Assume that $f \in I(K_3)$ and substitute $x = e_{12} - e_{21}$, $y = e_{23} - e_{32}$. This yields $\alpha_3 = 0 = \alpha_1$. As in the previous case, letting $y = x$ yields $\alpha_2 = 0$. Thus K_3 has no identity of type [2,2] which is symmetric in its variables.

An arbitrary polynomial of type [2,1,1] which is alternating in its degree 1 variables can be written (recall the notation (2.24))

$$\begin{aligned} f(x_1,x_2,y) = \alpha_1 y^2[x_1,\ x_2] + \alpha_2 y < x_1\ y\ x_2 > +\alpha_3 y[x_1,\ x_2]y \\ + \alpha_4 < x_1\ y\ x_2 > y + \alpha_5[x_1,\ x_2]y^2 + \alpha_6 < x_1\ y^2\ x_2 > . \end{aligned}$$

Assume that $f \in I(K_3)$. We wish to simplify f by adding known identities to it. By (2.11), subtracting a multiple of $q(x_1,x_2,y,y)$, we may assume that $\alpha_6 = 0$. Since identities of K_3 of type [3,1] have no terms of the form x_1y^3 or y^3x_1, letting $x_2 = y$ yields

$$\alpha_1 + \alpha_2 = 0, \qquad \alpha_4 + \alpha_5 = 0,$$

and

$$f(x_1,y,y) = (\alpha_2 + \alpha_3 - \alpha_5)[yx_1y,\ y].$$

By (2.12), subtracting a multiple of $p(y,y,x_1,y)$ from $f(x_1,y,y)$, we may assume that

$$\alpha_2 + \alpha_3 - \alpha_5 = 0$$

and

$$f(x_1,x_2,y)=\alpha_1(y^2[x_1,\ x_2]-y<x_1\ y\ x_2>)+(\alpha_1-\alpha_4)y[x_1,\ x_2]y$$
$$+\alpha_4(<x_1\ y\ x_2>y-[x_1,\ x_2]y^2).$$

By (2.8), adding $\alpha_4 r(x_1,x_2,y,y)$, we may assume that

$$f(x_1,x_2,y)=\alpha(y^2[x_1,\ x_2]-y<x_1\ y\ x_2>+y[x_1,\ x_2]y).$$

Finally, substituting $x_1=e_{12}-e_{21}$, $x_2=e_{23}-e_{32}$ and $y=e_{13}-e_{31}$ yields $\alpha=0$. Hence f is a consequence of p, q and r.

We now turn to the case $s=1$ and examine (3,4,1)-identities. Once again, we use (3.1) and concentrate our efforts on homogeneous polynomials of degree 4 of various types. Here the types [4] and [2,2] aren't relevant. A typical polynomial of type [3,1] has the form

$$f(x,y)=\alpha_1x^3y+\alpha_2x^2yx+\alpha_3xyx^2+\alpha_4yx^3$$

and we let $x\in K_3$, $y\in H_3$. Setting $x=e_{12}-e_{21}$, $y=e_{22}$ yields $\alpha_1+\alpha_3=0=\alpha_2+\alpha_4$, while the substitution $x=e_{12}-e_{21}$, $y=e_{23}-e_{32}$ implies $\alpha_1=\alpha_4=0$. Therefore, $f\equiv 0$.

Next we let

$$f(y,x_1,x_2)=\alpha_1y^2x_1x_2+\alpha_2y^2x_2x_1+\alpha_3yx_1yx_2+\alpha_4yx_2yx_1+\alpha_5yx_1x_2y$$
$$+\alpha_6yx_2x_1y+\alpha_7x_1yx_2y+\alpha_8x_2yx_1y+\alpha_9x_1x_2y^2+\alpha_{10}x_2x_1y^2$$
$$+\alpha_{11}x_1y^2x_2+\alpha_{12}x_2y^2x_1$$

be an arbitrary (3,4,1)-identity of type [2,1,1], $y,x_1\in K_3$, $x_2\in H_3$. We use the following four substitutions:

$$x_1=e_{12}-e_{21},x_2=e_{11},y=e_{23}-e_{32}\ \Rightarrow\alpha_1=\alpha_{10}=0,$$
$$x_1=e_{23}-e_{32},x_2=e_{11},y=e_{12}-e_{21}\ \Rightarrow\alpha_4=\alpha_7=0,$$
$$x_1=e_{12}-e_{21},x_2=e_{22},y=e_{23}-e_{32}\ \Rightarrow\alpha_2=-\alpha_{12}\text{ and }\alpha_9=-\alpha_{11},$$
$$x_1=e_{23}-e_{32},x_2=e_{23}+e_{32},y=e_{12}-e_{21}\ \Rightarrow\alpha_5=\alpha_6,\alpha_2=\alpha_9,$$
$$\text{and }\alpha_{11}=\alpha_{12}.$$

Putting all these together, we conclude that f has the form

$$f(y,x_1,x_2) = \alpha(y^2x_2x_1 + x_1x_2y^2 - x_1y^2x_2 - x_2y^2x_1) + \beta yx_1yx_2 + \gamma(yx_1x_2y + yx_2x_1y) + \delta x_2yx_1y.$$

Since there are no identities of type [3,1], the equation $f(y,y,x_2)=0$ gives $\alpha=\beta=-\gamma=\delta$; we then observe that $f(y,x_1,x_2)$ is a scalar multiple of $g(y,x_1,y,x_2)$.

Finally, by (3.1) we may assume that a (3,4,1)-identity of type [1,1,1,1] has the form

$$f(x_1,x_2,x_3,x_4) = \sum_{\sigma\in\mathcal{S}_3}(-1)^\sigma(\alpha x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}y + \beta x_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)} + \gamma x_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)} + \delta yx_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)})$$

with $x_i \in K_3$, $y\in H_3$. Letting $y=e_{11}+e_{22}+e_{33}$, we get $\alpha+\beta+\gamma+\delta=0$, and the substitution $x_1=e_{12}-e_{21}$, $x_2=e_{23}-e_{32}$, $x_3=e_{13}-e_{31}$, $y=e_{11}$ provides the relations $\delta=-\alpha$ and $\gamma=-\beta$ (char $F\neq 2$). Thus,

$$f(x_1,x_2,x_3,x_4) = \alpha r(x_1,x_2,x_3,x_4) + \beta[r(x_1,x_2,x_3,x_4) - \mathrm{S}_4(x_1,x_2,x_3,x_4)].$$

Now if we let

$$h(x_1,x_2,x_3,x_4) = \sum_{(123)} g(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)}),$$

it suffices to notice that $h-r=r-\mathrm{S}_4$ to conclude that f is a consequence of g and r. We have established

Theorem 3.5. *Under the hypotheses of (3.1) on the field F, any polynomial identity of $K_3(F,t)$ of minimal degree is a consequence of*

$$p(x_1,x_2,x_3,x_4) = \sum_{(123)}\{x_1\ [x_2,\ x_4]\ x_3\},$$
$$q(x_1,x_2,x_3,x_4) = \sum_{(123)}\{x_1\ [x_2,\ x_3]\ x_4\} + \sum_{(124)}\{x_1\ [x_2,\ x_4]\ x_3\} + 2([x_1,\ x_3]\mathrm{D}_{x_2,x_4} + [x_1,\ x_4]\mathrm{D}_{x_2,x_3} - [x_2,\ x_3]\mathrm{D}_{x_1,x_4} - [x_2,\ x_4]\mathrm{D}_{x_1,x_3}),$$
$$r(x_1,x_2,x_3,x_4) = [\mathrm{S}_3(x_1,x_2,x_3),\ x_4],$$

and any $*$-polynomial identity of $(M_3(F),t)$ of minimal degree is a consequence of

$$p(x_1-x_1^*,x_2-x_2^*,x_3-x_3^*,x_4-x_4^*),\ q(x_1-x_1^*,x_2-x_2^*,x_3-x_3^*,x_4-x_4^*),$$
$$r(x_1-x_1^*,x_2-x_2^*,x_3-x_3^*,x_4\pm x_4^*),\ g(x_1-x_1^*,x_2-x_2^*,x_3-x_3^*,x_4+x_4^*),$$

where $g(x_1,x_2,x_3,x_4)=\sum_{\sigma\in\mathcal{S}_2}(-1)^\sigma(\{x_{\sigma(1)}\ x_{\sigma(2)}\ (x_3\circ x_4)\}$. □

We move on to $n=4$. The following lemma shows that the only (4,5,s)-identities are the identities of K_4.

Lemma 3.6. *Over a field of characteristic not 2, $(M_n(F),t)$ has no $(n,n+1,1)$-identities when $n>3$.*

Proof. Write $f(x_1,\ldots,x_{n+1})=B_1+B_2+\ldots+B_n+B_{n+1}$, where

$$B_i=\sum_{\sigma\in\mathcal{S}_n}\alpha_\sigma^i x_{\sigma(1)}\ldots x_{\sigma(i-1)}x_{n+1}x_{\sigma(i)}\ldots x_{\sigma(n)},\qquad \alpha_\sigma^i\in F.$$

First observe that if we can show by way of substitutions that B_i is the zero polynomial for $1\le i\le[(n+1)/2]$ (where $[(n+1)/2]$ is the largest integer in $(n+1)/2$), then taking the transpose of these substitutions yields B_{n+2-i} is identically zero; therefore it suffices to establish that $B_i\equiv 0$ for $i=1,\ldots,[(n+1)/2]$, and for $i=(n+2)/2$, when n is even.

We first handle the case $n\ge 6$. Fix $i\in\{1,2,\ldots,[(n+1)/2]\}$. Substituting $(e_{12}-e_{21})$, $(e_{23}-e_{32})$, …, $(e_{i-1i}-e_{ii-1})$, e_{ii}, $(e_{ii+1}-e_{i+1i})$, …, $(e_{n-3n-2}-e_{n-2n-3})$, for $x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(i-1)},x_{n+1},x_{\sigma(i)},\ldots,x_{\sigma(n-3)}$, while successively choosing

1) $(e_{n-2n-1}-e_{n-1n-2}),(e_{n-1n}-e_{nn-1}),(e_{nn-1}-e_{n-1n})$
2) $(e_{n-2n-1}-e_{n-1n-2}),(e_{n-1n-2}-e_{n-2n-1}),(e_{n-2n}-e_{nn-2})$
3) $(e_{n-2n-1}-e_{n-1n-2}),(e_{n-1n}-e_{nn-1}),(e_{nn-2}-e_{n-2n})$

for $x_{\sigma(n-2)}$, $x_{\sigma(n-1)}$, $x_{\sigma(n)}$, yields

$$\alpha_\sigma^i=-\alpha_{\sigma\tau_1}^i=-\alpha_{\sigma\tau_2}^i=\alpha_{\sigma\tau_3}^i,$$

where $\tau_1=(n-1\ n)$, $\tau_2=(n-2\ n-1)$ and $\tau_3=(n-2\ n)$, $\sigma\in\mathcal{S}_n$. Since $\tau_1=\tau_2\tau_3\tau_2$, we have $\alpha_\sigma^i=0$. If n is odd, we now have $f\equiv 0$; if

n is even, we are left with $f = B_k$, $k = n+2/2$. But the substitution $(e_{12} - e_{21})$, …, $(e_{k-1k} - e_{kk-1})$, e_{kk}, $(e_{kk-2} - e_{k-2k})$, $(e_{k-2k+1} - e_{k+1k-2})$, $(e_{k+1k+2} - e_{k+2k+1})$, …, $(e_{n-1n} - e_{nn-1})$, for $x_{\sigma(1)}, \ldots, x_{\sigma(k-1)}, x_{n+1}, x_{\sigma(k)}, x_{\sigma(k+1)}, x_{\sigma(k+2)}, \ldots, x_{\sigma(n)}$ forces $\alpha_\sigma^k = \alpha_{\sigma\tau}^{k-1} = 0$, where $\tau = (k-2\ k)$, so again $f \equiv 0$.

The cases $n = 4$ or 5 are handled in the same way except for $i = 2$ when $n = 4$ and $i = 3$ when $n = 5$. For those, substitutions 2) and 3) should be replaced by

2′) $(e_{12} - e_{21}), (e_{23} + e_{32}), (e_{34} - e_{43}), (e_{43} - e_{34}), (e_{32} - e_{23})$

2″) $(e_{12} - e_{21}), (e_{23} - e_{32}), (e_{34} + e_{43}), (e_{45} - e_{54}), (e_{54} - e_{45}), (e_{43} - e_{34})$

which imply

$$\alpha_\sigma^i + \alpha_{\sigma\tau_1}^i - \alpha_{\sigma\tau_2}^{n+1} - \alpha_{\sigma\tau_3}^{n+1} = 0,$$

where $\tau_1 = (n-2\ n-1)$, $\tau_2 = (n-2\ n\ n-1)$ and $\tau_3 = (n-2\ n)$, and

3′) $(e_{12} - e_{21}), e_{22}, (e_{24} - e_{42}), (e_{43} - e_{34}), (e_{32} - e_{23})$

3″) $(e_{12} - e_{21}), (e_{23} - e_{32}), e_{33}, (e_{35} - e_{53}), (e_{54} - e_{45}), (e_{43} - e_{34})$

which in turn imply

$$\alpha_\sigma^i - \alpha_{\sigma\tau}^i + \alpha_\sigma^{n+1} - \alpha_{\sigma\tau}^{n+1} = 0,$$

where $\tau = (n-2\ n-1)$.

The substitution $(e_{12}-e_{21})$, $(e_{23}+e_{32})$, $(e_{34}-e_{43})$, $(e_{43}-e_{34})$, $(e_{32}-e_{23})$, for $x_{\sigma(1)}, x_5, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}$ implies $\alpha_\sigma^2 + \alpha_{\sigma\tau_1}^2 - \alpha_{\sigma\tau_2}^5 - \alpha_{\sigma\tau_3}^5 = 0$, where $\tau_1 = (23)$, $\tau_2 = (243)$ and $\tau_3 = (24)$; thus $\alpha_\sigma^2 = -\alpha_{\sigma\tau_1}^2$. Finally, the substitution $(e_{12} - e_{21})$, $(e_{23} - e_{32})$, $(e_{34} - e_{43})$, $(e_{45} - e_{54})$, $(e_{54} - e_{45})$, $(e_{43} - e_{34})$ for $x_{\sigma(1)}, x_{\sigma(2)}, x_6, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}$ yields $\alpha_\sigma^3 + \alpha_{\sigma\tau_1}^3 - \alpha_{\sigma\tau_2}^6 - \alpha_{\sigma\tau_3}^6 = 0$, where $\tau_1 = (34)$, $\tau_2 = (354)$ and $\tau_3 = (35)$; so $\alpha_\sigma^2 = -\alpha_{\sigma\tau_1}^2$. □

We now show

Theorem 3.7. *Under the hypotheses of (3.1) on the field F, any polynomial identity of $K_4(F,t)$ of minimal degree is a consequence of*

$$\kappa(x_1, x_2, x_3, y) := y^2 \mathrm{S}_3(\mathrm{V}_{x_1}, \mathrm{V}_{x_2}, \mathrm{V}_{x_3}) - (y\mathrm{S}_3(\mathrm{V}_{x_1}, \mathrm{V}_{x_2}, \mathrm{V}_{x_3})) \circ y,$$

and any $*$-polynomial identity of $(M_4(F),t)$ of minimal degree is a consequence of $\kappa(x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*, y - y^*)$.

Proof. It is obvious that $f(x) = x^5$ isn't an identity, and by Kostant-Rowen, K_4 doe not satisfy the standard polynomial S_5. This takes care of types [5] and [1,1,1,1,1]. Consider now an arbitrary polynomial of type [4,1]

$$f(x,y) = \alpha_1 x^4 y + \alpha_2 x^3 yx + \alpha_3 x^2 yx^2 + \alpha_4 xyx^3 + \alpha_5 yx^4.$$

The coefficients of e_{13}, e_{31} and e_{12} in $f(e_{12} - e_{21}, e_{12} - e_{21} + e_{13} - e_{31} + e_{14} - e_{41})$ yield $\alpha_1 = \alpha_5 = 0$ and $\alpha_2 + \alpha_3 + \alpha_4 = 0$; then the coefficients of e_{12}, e_{14} and e_{21} in $f(e_{12} - e_{21} + e_{23} - e_{32} + e_{24} - e_{42}, e_{14} - e_{41})$ lead to $\alpha_2 = \alpha_3 = \alpha_4 = 0$.

Next, an arbitrary (4,5,0)-identity of type [3,2] would have the form

$$\begin{aligned} f(x,y) =& \alpha_1 y^2 x^3 + \alpha_2 yxyx^2 + \alpha_3 yx^2 yx + \alpha_4 yx^3 y + \alpha_5 xy^2 x^2 \\ &+ \alpha_6 xyxyx + \alpha_7 xyx^2 y + \alpha_8 x^2 y^2 x + \alpha_9 x^2 yxy + \alpha_{10} x^3 y^2. \end{aligned}$$

We successively compute

$$\begin{aligned} f(e_{12} - &e_{21} + e_{13} - e_{31}, e_{24} - e_{42}) = (\alpha_5 + 2\alpha_{10})e_{12} + \alpha_5 e_{13} \\ &+ (-2\alpha_1 - \alpha_8)e_{21} - \alpha_8 e_{31} \end{aligned}$$

which produces $\alpha_1 = \alpha_5 = \alpha_8 = \alpha_{10} = 0$, then

$$\begin{aligned} f(e_{13} - &e_{31} + e_{24} - e_{42}, e_{12} - e_{21}) = (\alpha_1 + \alpha_3 + \alpha_8)(e_{13} + e_{24}) \\ &+ (-\alpha_5 - \alpha_7 - \alpha_{10})(e_{31} + e_{42}) \end{aligned}$$

which yields $\alpha_3 = \alpha_7 = 0$, and finally

$$\begin{aligned} f(e_{13} - e_{31} + e_{23} - e_{32}, e_{13} - e_{31} + e_{14} - e_{41}) &= (4\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 \\ &+ 2\alpha_5 + \alpha_6 + \alpha_7 + 2\alpha_8 + \alpha_9 + 2\alpha_{10})e_{13} + (2\alpha_4 + \alpha_7 + \alpha_9 + 2\alpha_{10})e_{14} \\ &+ (2\alpha_5 + \alpha_6 + \alpha_7 + 2\alpha_8 + \alpha_9 + 2\alpha_{10})e_{23} + (\alpha_7 + \alpha_9 + 2\alpha_{10})e_{24} \\ &- (2\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 + 2\alpha_8 + 2\alpha_9 + 4\alpha_{10})e_{31} \\ &- (2\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_5 + \alpha_6 + 2\alpha_8)e_{32} - (2\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4)e_{41} \\ &- (2\alpha_1 + \alpha_2 + \alpha_3)e_{42} \end{aligned}$$

in which the coefficients of e_{42}, e_{24}, e_{23} and e_{41} imply that $\alpha_2 = \alpha_9 = \alpha_6 = \alpha_4 = 0$. This shows that K_4 has no identity of type [3,2].

Next, we treat the case [3,1,1]. By (3.1), it suffices to look at polynomials of the form

$$\begin{aligned} f(x, y_1,y_2) = {} & \alpha_1(x^3y_1y_2 - x^3y_2y_1) + \alpha_2(x^2y_1xy_2 - x^2y_2xy_1) \\ & + \alpha_3(x^2y_1y_2x - x^2y_2y_1x) + \alpha_4(xy_1x^2y_2 - xy_2x^2y_1) \\ & + \alpha_5(xy_1xy_2x - xy_2xy_1x) + \alpha_6(xy_1y_2x^2 - xy_2y_1x^2) \\ & + \alpha_7(y_1x^3y_2 - y_2x^3y_1) + \alpha_8(y_1x^2y_2x - y_2x^2y_1x) \\ & + \alpha_9(y_1xy_2x^2 - y_2xy_1x^2) + \alpha_{10}(y_1y_2x^3 - y_2y_1x^3). \end{aligned}$$

Once again, we use three substitutions:

$$f(e_{12} - e_{21}, e_{23} - e_{32}, e_{34} - e_{43}) = -\alpha_1e_{14} - \alpha_{10}e_{41}$$

from which we deduce that $\alpha_1 = \alpha_{10} = 0$,

$$\begin{aligned} f(e_{12}-e_{21} + e_{13} - e_{31}, e_{12} - e_{21}, e_{23} - e_{32}) = {} & 2(\alpha_1 + \alpha_3 \\ & + \alpha_6 + \alpha_{10})e_{11} - (\alpha_2 - \alpha_3 + \alpha_4 - \alpha_6 + 4\alpha_7 + \alpha_8 + \alpha_9)e_{22} \\ & + (2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + 2\alpha_7 - \alpha_8 - \alpha_9)e_{23} \\ & + (-\alpha_2 + \alpha_3 - \alpha_4 + \alpha_6 + 2\alpha_7 + \alpha_8 + \alpha_9 + 2\alpha_{10})e_{32} \\ & + (2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8 + \alpha_9 + 2\alpha_{10})e_{33} \end{aligned}$$

which implies that $\alpha_7 = 0$, $\alpha_8 = -\alpha_9$ and $\alpha_2 = -\alpha_4$, and also

$$\begin{aligned} f(e_{12} - e_{21} + e_{34} - e_{43}, e_{23} - e_{32} + e_{24} - e_{42} + e_{34} - e_{43}, e_{13} - e_{31}) = {} & \\ (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8 + \alpha_{10})(e_{11} + e_{22}) - (\alpha_2 - \alpha_5 + \alpha_7 + \alpha_9)(e_{12} & \\ + e_{21} + e_{34} + e_{43}) - (\alpha_2 + \alpha_3 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10})e_{13} & \\ - (\alpha_1 + \alpha_4 + \alpha_5 + \alpha_6)e_{24} - (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_9)e_{31} & \\ - 2(\alpha_2 + \alpha_7 + \alpha_9)e_{33} - (\alpha_3 + \alpha_5 + \alpha_8 + \alpha_{10})e_{42} - 2\alpha_5e_{44} & \end{aligned}$$

which implies that $\alpha_4 = -\alpha_8$, $\alpha_2 = -\alpha_9$, $\alpha_5 = 0$ and $\alpha_4 = -\alpha_6$. These relations then force f to be a scalar multiple of the polynomial

$$
\begin{aligned}
\tilde{f}(x,y,z) &= x^2(yz-zy)x - x(yz-zy)x^2 - x^2yxz + x^2zxy + xyx^2z \\
&\quad - xzx^2y - yx^2zx + zx^2yx + yxzx^2 - zxyx^2 \\
&= (\{xyz\} - \{xzy\})U_x + zU_xU_{x,y} - \{yxzU_x\} - yU_xU_{x,z} \\
&\quad + \{zxyU_x\}.
\end{aligned}
$$

Now it suffices to notice from (2.23) that $\tilde{f}(x,y,z) = \kappa(z,y,x,x)$, that is, f is a consequence of our polynomial κ.

We continue with an arbitrary polynomial of type [2,2,1]; using (3.1), it has the form

$$
\begin{aligned}
f(x,y,z) &= \alpha_1(x^2y^2z + y^2x^2z) + \alpha_2(x^2yzy + y^2xzx) + \alpha_3(x^2zy^2 + y^2zx^2) \\
&+ \alpha_4(xzxy^2 + yzyx^2) + \alpha_5(zx^2y^2 + zy^2x^2) + \alpha_6(xyxyz + yxyxz) \\
&+ \alpha_7(xyxzy + yxyzx) + \alpha_8(xyzxy + yxzyx) + \alpha_9(xzyxy + yzxyx) \\
&+ \alpha_{10}(zxyxy + zyxyx).
\end{aligned}
$$

We first compute the substitution

$$
\begin{aligned}
f(e_{12}-e_{21}, e_{13}-e_{31}, e_{12}-e_{21}+e_{13}-e_{31}+e_{14}-e_{41}) &= 2\alpha_1 e_{14} - 2\alpha_5 e_{41} \\
+ (2\alpha_1+\alpha_2+\alpha_3)e_{12} - (\alpha_3+\alpha_4+2\alpha_5)e_{21} + \dots
\end{aligned}
$$

from which we get $\alpha_1 = \alpha_5 = 0,$ and $\alpha_2 = \alpha_4 = -\alpha_3$. Then the substitution

$$
\begin{aligned}
&f(e_{12}-e_{21}+e_{13}-e_{31}+e_{14}-e_{41}, e_{12}-e_{21}+e_{34}-e_{43}, e_{23}-e_{32}) = \\
&\quad -(\alpha_4+\alpha_6+\alpha_8+\alpha_9)e_{12} - (\alpha_4+\alpha_9)e_{13} - (3\alpha_2+\alpha_4+\alpha_7-\alpha_9)e_{14} \\
&\quad + (\alpha_2+\alpha_7+\alpha_8+\alpha_{10})e_{21} - (2\alpha_1-2\alpha_5+\alpha_6-\alpha_{10})e_{22} \\
&\quad + (2\alpha_1+2\alpha_3+2\alpha_5+2\alpha_6-\alpha_8)e_{23} + (\alpha_3+2\alpha_5-\alpha_8)e_{24} \\
&\quad + (\alpha_2+\alpha_7)e_{31} - (2\alpha_1+2\alpha_3+2\alpha_5-\alpha_8+2\alpha_{10})e_{32} \\
&\quad + (2\alpha_1-2\alpha_5+\alpha_6-\alpha_{10})e_{33} - (\alpha_3+2\alpha_5+\alpha_{10})e_{34} \\
&\quad + (\alpha_2+3\alpha_4-\alpha_7+\alpha_9)e_{41} - (2\alpha_1+\alpha_3-\alpha_8)e_{42} \\
&\quad + (2\alpha_1+\alpha_3+\alpha_6)e_{43}
\end{aligned}
$$

implies that all remaining α_i's must vanish. Therefore, there are no identities of type [2,2,1].

The only remaining case is type [2,1,1,1]. A typical polynomial looks like

$$f(x_1,x_2,x_3,y) = \sum_{\sigma\in\mathcal{S}_3} (\alpha_1 x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}y^2 + \alpha_2 x_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)}y \\ + \alpha_3 x_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)}y + \alpha_4 yx_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}y + \alpha_5 x_{\sigma(1)}x_{\sigma(2)}y^2x_{\sigma(3)} \\ + \alpha_6 x_{\sigma(1)}yx_{\sigma(2)}yx_{\sigma(3)} + \alpha_7 yx_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)} + \alpha_8 x_{\sigma(1)}y^2x_{\sigma(2)}x_{\sigma(3)} \\ + \alpha_9 yx_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)} + \alpha_{10}y^2x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}).$$

From the substitution

$$f(e_{12}-e_{21}+e_{13}-e_{31}, e_{23}-e_{32}, e_{34}-e_{43}, e_{12}-e_{21}+e_{23}-e_{32}+e_{34} \\ - e_{43}) = (\alpha_2+\alpha_3+\alpha_7+\alpha_9)e_{11} - (\alpha_5+\alpha_6-\alpha_9-\alpha_{10})(e_{12}+e_{43}) \\ - (\alpha_1+\alpha_2+2\alpha_5+\alpha_6+2\alpha_8+\alpha_9+\alpha_{10})(e_{14}+e_{41}) + (\alpha_1+\alpha_2 \\ - \alpha_6-\alpha_8)(e_{21}+e_{34}) + (\alpha_2+\alpha_3+\alpha_7+\alpha_9)e_{22} + (\alpha_3+\alpha_4+\alpha_6 \\ + \alpha_7)(e_{23}+e_{32}) - (\alpha_2-\alpha_3+\alpha_5-\alpha_8)e_{24} + (2\alpha_3+4\alpha_6+2\alpha_7)e_{33} \\ + (\alpha_5+\alpha_7-\alpha_8-\alpha_9)e_{42} - 2(\alpha_2+\alpha_9)e_{44}$$

we deduce the relations $\alpha_1 = -\alpha_{10}$, $\alpha_2 = -\alpha_9$, $\alpha_3 = -\alpha_7$, $\alpha_4 = -\alpha_6$, $\alpha_5 = -\alpha_8$, $\alpha_6 = 0$, $\alpha_1+\alpha_2-\alpha_8 = 0$ and $\alpha_1+\alpha_3+\alpha_8 = 0$. The dimension of the nullspace is 2 and we let α_1 and α_8 play the role of free variables. We first set $\alpha_1 = \alpha_8 = 1$: then $\alpha_2 = \alpha_4 = \alpha_6 = \alpha_9 = 0$, $\alpha_3 = -2$, $\alpha_5 = -1$, $\alpha_7 = 2$, $\alpha_{10} = -1$ and we get

$$f_1(x_1,x_2,x_3,y) = \sum_{\sigma\in\mathcal{S}_3} (-1)^\sigma (x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}y^2 - 2(x_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)}y \\ - yx_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)}) - x_{\sigma(1)}x_{\sigma(2)}y^2x_{\sigma(3)} + x_{\sigma(1)}y^2x_{\sigma(2)}x_{\sigma(3)} \\ - y^2x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)});$$

but it is easily seen that $f_1(x_1,x_2,y,y) = \kappa(x_1,x_2,y,y) - 2q(y,x_1,x_2)$. Next we let $\alpha_1 = 0$, $\alpha_8 = 1$: then $\alpha_4 = \alpha_6 = \alpha_{10} = 0$ while $\alpha_3 = \alpha_5 = \alpha_9 = -1$ and $\alpha_2 = \alpha_7 = 1$ and we get

$$f_2(x_1,x_2,x_3,y) = \sum_{\sigma\in\mathcal{S}_3} (-1)^\sigma (x_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)}y \\ - x_{\sigma(1)}x_{\sigma(2)}y^2x_{\sigma(3)} - x_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)}y + x_{\sigma(1)}y^2x_{\sigma(2)}x_{\sigma(3)} \\ + yx_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)} - yx_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)})$$

and from (2.23), $f_2(x_1, x_2, x_3, y) = \kappa(x_1, x_2, x_3, y)$.

Thus, any [2,1,1,1]-identity and in fact any identity of $K_4(F)$ is a consequence of our polynomial κ. □

In a subsequent paper, we will present the minimal degree $*$-identities of $(M_n(F), s)$, $n < 5$, for s the symplectic involution. For the transpose involution, the minimal identities involved mostly skew-symmetric variables; naturally, when $* = s$, it appears that most variables will be symmetric. But regardless of the involution, we do not expect the surprising richness of $*$-identities that occurred for low n's to carry over to the general case $n \geq 5$.

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