

The Structure of Quadratic Jordan Systems of Clifford Type

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Dedicated to the memory of Gottfried Ruettimann

0. Introduction

Zel'manov's classification of prime nondegenerate linear Jordan triple systems and pairs parallels in many ways the corresponding structure theory for Jordan algebras [26] in the sense that both theories follow the same basic structural flowchart. In [29], Zel'manov showed that prime nondegenerate quadratic Jordan systems (like their algebra counter-parts) are either *i*-special, or exceptional finite-dimensional over their centroid, and further established that in characteristic $\neq 2,3$, the *i*-special triples (resp. pairs) are forms of either one of 5 simple triples (resp. 2 pairs) of hermitian type or one of 4 simple triples (resp. 2 pairs) of Clifford type. (As it turns out, he inadvertently omitted two pairs—hence four triples— of the latter type.) The part of the argument that pertains to systems of hermitian type has been extended to quadratic Jordan systems over an arbitrary ring of scalars [6]; in the present work, we finalize the general classification by handling the case of quadratic systems of Clifford type.

By system of Clifford type (triple or pair), we mean a system all the

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homotopes of which strictly satisfy some Jordan Clifford identity, a Jordan polynomial that does not vanish on the split 3×3 hermitian matrices $H(M_3(\Phi), t)$. [The dichotomy in the structure of simple i-special systems depends on the existence of Jordan PI's which have both a Clifford and "hermitian" heritage, the so-called **Zel'manov polynomials** ; the construction of one such creature is carried out in [5] for quadratic systems.] Our basic philosophy consists in reducing the general setting of simple systems to systems of finite capacity.

In section 2, we first narrow down Loos' list of **classical** Jordan central simple pairs of finite capacity to the three kinds of i-special pairs capable of satisfying a Clifford homotope identity: the rectangular pair $\mathcal{V}(M, R, f)$ (see (1.11.2)), the Clifford pair $J(q)$ of a quadratic form q (see (1.9–10)) and the alternating pair $A_5(\Phi)$ (see (1.7–8)). This involves a scrutiny of small division pairs (section 2) and rectangular pairs (section 3).

In section 4, we reveal what perhaps ought to be considered the true nature of a "degree 2" Jordan system: the local algebras of each of our Cliffordian candidates are Jordan algebras of degree 2 or less. The usefulness of this information is best illustrated by the Primitive Form Theorem (section 5): it essentially asserts that if a central simple pair V is a scalar form of a pair \bar{V} (over a field) all of whose local algebras are Clifford algebras, then V must be semiprimitive (it contains nonzero regular elements), hence primitive. We end the preparatory work in section 6 with the verification that some Clifford homotope identity does indeed vanish on each member of our list, so all of them are bonafide pairs of Clifford type.

We devote section 7 to piecing together the structure of primitive pairs. Most of the ground work was laid out in [7]: a primitive system J with a (homotope) PI necessarily has nonzero socle, and if the PI is Clifford, then $J = Soc(J)$ is actually simple of strict finite capacity ≤ 2 . Section 8 establishes the pivotal fact that for a system with a polynomial identity over a large uncountable field, nondegeneracy and semiprimitivity are equivalent. This allows us to handle the delicate issue of scalar extensions in characteristic 2: given a simple pair V of Clifford type, we take a TIGHT scalar extension V_Ω over a suitably large field so that V_Ω is now semiprimitive (since V_Ω remains simple (by tightness) and inherits the Clifford homotope identity); hence V_Ω is primitive classical of capacity ≤ 2 and by the Primitive Form Theorem, the same holds for V . The structure of simple triples then emerges from their categorical equivalence with Jordan pairs with involution

(see section 9).

Finally, section 10 of the paper explains why a prime nondegenerate system sits inside a simple one as a scalar form. Over a field, a locally finite-dimensional idempotent-finite Jordan system with no nonzero locally-nilpotent ideals is a direct sum of finitely many simple ideals the nature of which is known from sections 8 and 9. We combine this with an ultrafilter argument that consists in embedding (as a scalar form) the strongly prime J in some ultraproduct \tilde{J} that taps into the common algebraic features of its simple factors; by primeness, J must lie completely within one of \tilde{J} 's simple building blocks.

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1. Review

We refer the reader to [13] for most of the basic material about quadratic Jordan systems found below. Unless specified otherwise, we take Φ to be an arbitrary ring of scalars. A **Jordan triple system** J over Φ is a Φ -module equipped with a product $P_x y$ quadratic in x and linear in y (where the linearization $P_{x,z} := P_{x+z} - P_x - P_z$ yields $P_{x,z}(y) =: \{xyz\} =: L_{x,y}(z)$) strictly satisfying the identities

$$(J1) \quad L_{x,y}P_x = P_x L_{y,x} = P_{P(x)y,x}$$

$$(J2) \quad L_{x,P(y)x} = L_{P(x)y,y}$$

$$(J3) \quad P_{P(x)y} = P_x P_y P_x$$

(hence (J1)-(J3) continue to hold in all scalar extensions of J).

In a **unital Jordan algebra** (a triple plus a choice of element 1 with $P_1 = Id$), we write $U_x y$ and $V_{x,y} z$ for $P_x y$ and $L_{x,y} z$ respectively. By a **Jordan algebra**, we mean a subspace of some unital Jordan algebra closed under $U_x y$ and the **square** $x^2 := U_x 1$. A **Jordan pair** $V = (V^+, V^-)$ is a pair of Φ -modules acting on each other in a Jordan triple fashion; we traditionally denote its products by $Q_{x^\varepsilon} y^{-\varepsilon}$ and $D_{x^\varepsilon, y^{-\varepsilon}} z^\varepsilon$, $\varepsilon = \pm$. We will rely heavily on the well-known categorical equivalences between triples and pairs: Jordan pairs are equivalent to **polarized Jordan triples** via the

polarizing and de-polarizing functors

$$(1.1.1) \quad V = (V^+, V^-) \implies Pol(V) = V^+ \oplus V^- : \\ P_{x^+ \oplus x^-}(y^+ \oplus y^-) = Q_{x^+}y^- \oplus Q_{x^-}y^+,$$

$$(1.1.2) \quad J = J^+ \oplus J^- \implies \mathcal{V}(J, pol) = (J^+, J^-).$$

Here a polarized J is simple as triple iff it is simple as polarized triple (no polarized ideals) iff V is simple as pair. (Note that if I is a triple ideal, then $\underline{I} \subset I \subset \bar{I}$ for polarized ideals $\underline{I}^\varepsilon = I \cap V^\varepsilon$, $\bar{I}^\varepsilon = \pi_\varepsilon(I)$. If $\underline{I} = 0$, $\bar{I} = J$, then $P_J J = 0$ since $P_I J + P_J I + \{JJI\} \subset \underline{I}$. The polarized ideals $I^+ \oplus I^-$ of J are precisely the pair ideals (I^+, I^-) of V .)

Ordinary Jordan triples J are equivalent to **Jordan pairs** (V, η) with **involution** (i.e. a pair antiautomorphism $\eta_\varepsilon : V^\varepsilon \rightarrow V^{-\varepsilon}$ such that $\eta_{-\varepsilon} \circ \eta_\varepsilon = Id$ on V^ε , $\varepsilon = \pm$) via

$$J \text{ a triple} \implies (\mathcal{V}(J), \text{ex}) \text{ is a pair with involution for} \\ (1.2.1) \quad \mathcal{V}(J) = (J, J) \text{ under } Q_{x^\varepsilon}y^{-\varepsilon} = P_x y \text{ and exchange involution} \\ (x, y)^{\text{ex}} = (y, x), \text{ (the ex-invariant pair ideals of } \mathcal{V}(J) \text{ are} \\ \text{precisely all } \mathcal{V}(I) = (I, I) \text{ for triple ideals } I \triangleleft J),$$

$$(V, \eta) \text{ a pair with involution} \implies \mathcal{J}(V, \eta) = H(Pol(V), \eta) \\ (1.2.2) \quad \text{is a triple isomorphic to the isotope } V^{+(\eta_+)} : \\ P_{x^+}y^+ = Q_{x^+}[(y^+)^{\eta_+}].$$

$$(1.2.3) \quad \text{These constructions are inverses: } \mathcal{J}(\mathcal{V}(J), \text{ex}) \simeq J, \\ (\mathcal{V}(\mathcal{J}(V, \eta)), \text{ex}) \simeq (V, \eta).$$

If J and (V, η) correspond ($J \cong \mathcal{J}(V, \eta)$) and also

$$(1.2.4) \quad (V, \eta) \cong (\mathcal{V}(J), \text{ex}), \text{ then } J \text{ polarized} \iff V = W \boxplus W^\eta \\ \text{for } W \triangleleft V \text{ (in which case } J = Pol(W)).$$

(If $J = J^+ \oplus J^-$ then [identifying (V, η) with $(\mathcal{V}(J), ex)$] $V = W \oplus W^{ex}$ for $W := (J^+, J^-)$, $W^{ex} = (J^-, J^+)$ where W is an ideal [hence orthogonal to the ideal W^{ex}] by $Q_W V \subset (P_{J^+} J, P_{J^-} J) \subset (J^+, J^-)$ and $\{WVV\} \subset (\{J^+ J J\}, \{J^- J J\}) \subset (J^+, J^-)$. Conversely, if $V = W \boxplus W^\eta$ then [identifying J with $V^{+(\eta)}$] $J = J^+ \oplus J^-$ for $J^+ := W^+$, $J^- := (W^\eta)^+ = (W^-)^\eta$ where $J^+ + (J^-)^\eta = W$, $J^- + (J^+)^\eta = W^\eta$, so $P_{J^+} J \subset Q_{W^+} [V]^\eta \subset W^+ = J^+$, $P_{J^-} J \subset Q_{(W^-)^\eta} [V]^\eta = [Q_{W^-} V]^\eta \subset [W^-]^\eta = J^-$, $P_{J^+} J^+ \subset Q_W [W]^\eta = 0$, $P_{J^-} J^- \subset Q_{W^\eta} [W^\eta]^\eta = [Q_W W^\eta]^\eta = 0$, and $\{J^+ J J^-\} \subset \{WVW^\eta\} = 0$, hence J is polarized.)

Most of the important Jordan systems come from associative structures. Using the ambient product in an associative algebra (respectively triple or pair) A , we can Jordanify A in the usual way: either

$$(1.3.1) \quad A^J : P_x y = xyx,$$

or

$$(1.3.2) \quad A^{*J} : P_x y = xy^* x, \quad * \text{ an involution in } A.$$

(If $A = (A^+, A^-)$ is an associative pair, or a polarized associative triple system, we assume that $*$ is polarized, i.e. $(A^\varepsilon)^* \subset A^\varepsilon$). We say a Jordan system is **special** if it is a subsystem of some A^J (equivalently, of some A^{*J}).

For instance, examples of special triples include **hermitian triples**

$$(1.4) \quad H(A, *) = \{x \in A \mid x^* = x\} \subseteq A^{*J}, \quad P_x y = xy^* x = xyx,$$

and more generally **ample hermitian triples**

$$(1.5) \quad \begin{aligned} & H_\circ(A, *) \subseteq H(A, *), \quad aH_\circ a^* \subseteq H_\circ \text{ for all } a \in A, \text{ and all traces} \\ & a + a^* \text{ lie in } H_\circ; \text{ if } A \text{ is an algebra, then all the norms } aa^* \\ & \text{must also lie in } H_\circ \text{ (equivalently, for } A \text{ unital, } 1 \in H_\circ). \end{aligned}$$

H_\circ is always an outer ideal in H , and if $1/2 \in \Phi$ the only ample subspace is $H_\circ = H$. In particular, for the unital matrix algebra $M_n(\Delta)$, $-$ an involution of the coordinate algebra Δ , and $\Delta_\circ \subseteq H(\Delta, -)$ an ample subspace, we can form the algebra of hermitian matrices with diagonal entries in Δ_\circ , hence a triple

$$(1.6) \quad H_n(\Delta, \Delta_\circ) : P_x y = x\bar{y}^t x = xyx.$$

Another family of special triples are the **skew triples** $Sk(A, *) = \{x \in A \mid x^* = -x\}$, and more importantly the **alternating triples** consisting of all **skewtraces**

$$(1.7) \quad Alt(A, *) = \{a - a^* \mid a \in A\} \subseteq A^{*J}, P_x y = xy^*x = -xyx.$$

$Alt(A, *)$ is always an outer ideal in $Skew(A, *)$, and if $1/2 \in \Phi$ then all skew elements are skewtraces, and $Alt = Sk$. If $A = M_n(\Phi)$ and $*$ is the transpose involution, we recover the usual alternating matrices

$$(1.8.1) \quad A_n(\Phi) : P_x y = xy^t x = -xyx.$$

This description is tied to a particular basis. We can think of $M_n(\Phi)$ as $End(W)$ for $W = \Phi^n$ a finite-dimensional free module over Φ with inner product $\sigma(x, y)$ and transpose as the adjoint involution $*$ relative to σ . Then $A_n(\Phi) = Alt(W, \sigma)$ consists of the usual σ -alternating linear transformations on W ,

$$(1.8.2) \quad Alt(W, \sigma) = \{x \in End(W) \mid \sigma(x(w), w) = 0 \ \forall w \in W\}, \\ P_x y = xy^*x = -xyx$$

for any nondegenerate symmetric bilinear form σ . The Jordan pair $\mathcal{V}(Alt(W, \sigma))$ has an even more intrinsic description without reference to any σ (just the natural pairing between W and W^*): if we define a linear transformation $x : W \rightarrow W^*$ (and dually for $W^* \rightarrow W$) to be **alternating** if the image $x(w)$ of any w is a linear functional which kills its parent, $[x(w)](w) = 0$, then

$$(1.8.3) \quad \mathcal{V}(Alt(W, \sigma)) \simeq (Alt(W, W^*), Alt(W^*, W)) \text{ with } Q_x y = -xyx \\ (W \text{ a finite-dimensional free } \Phi\text{-module})$$

via the isomorphism $(x, y) \rightarrow (s \circ x, y \circ s^{-1})$ for $s : W \rightarrow W^*$ the linear isomorphism $s(w) = \sigma(w, \cdot) = \sigma(\cdot, w)$. (It is easy to check that $x \in Alt(W, \sigma) \Leftrightarrow s \circ x \in Alt(W, W^*) \Leftrightarrow$ (since σ is symmetric) $x \circ s^{-1} \in Alt(W^*, W)$.) We could do the same construction for any nondegenerately paired spaces W^+, W^- .

The *odd* alternating matrices $A_{2n+1}(\Phi)$ of (1.8.1) have no invertible elements, so cannot be reduced to Jordan algebras. However, the *even* alternating matrices $A_{2n}(\Phi)$ are just hermitian matrices disguised by isotopy

$$(1.8.4) \quad A_{2n}(\Phi) \cong H_n(Q(\Phi), \Phi)^{(\sigma)} \quad (\sigma(X) = -SXS)$$

and isotopic triples produce isomorphic duplicated pairs

$$(1.8.5) \quad \mathcal{V}(A_{2n}(\Phi)) \cong \mathcal{V}(H_n(Q(\Phi), \Phi))$$

under the explicit isomorphism $(X^+, X^-) \longrightarrow (SX^+, X^-S)$ (see (1.13.3) below for more details).[†]

As another example of a subtriple system of an A^{*J} , we have the triple of rectangular matrices $M_{p,q}(\Delta) \subseteq M_{p+q,p+q}(\Delta)^{*J}$ where $x^* = \bar{x}^t$, – an involution on Δ .

Many of the arguments below will revolve around the quintessential degree 2 system, the **Clifford triple $\mathbf{J}(\mathbf{q})$** built out of a quadratic form

$$(1.9) \quad J(q) : P_x y = q(x, y)x - q(x)y.$$

In characteristic 2, we may also take an algebra outer ideal I containing 1 in a Jordan Clifford algebra $J(q, 1)$,

$$(1.10.1) \quad J(q, 1) : U_x y = q(x, \bar{y})x - q(x)\bar{y}, \text{ where } q(1) = 1, \\ \bar{y} = t(y)1 - y \quad (t(y) = q(1, y)).$$

In Jordan Clifford algebras the bar is always an algebra involution,

[†] We thank the referee for pointing out that this isomorphism (1.8.5) at the pair level does not reach down to the triple level, $A_{2n}(\Phi) \not\cong H_n(Q(\Phi), \Phi)$ as triples when $n > 1$ (for $n = 1$ they are both $\cong \Phi^t$): the latter has unit $u = 1$, whereas the former has no unit. [If $P_u y = y$ for all $y \in A_{2n}$ then u is invertible, $-uyu = y$ forces $u^{-1} = -u$, $uyu^{-1} = y$, so u centralizes A_{2n} and hence the algebra M_{2n} it generates when $n \geq 2$, so $u \in \Phi 1 \cap A_{2n} = 0$, a contradiction. Or, directly, the alternating matrix u has entries $u_{ii} = 0$, and $0 = E_{ii}(y + yu)E_{kk}$ imply for $y = E_{ij} - E_{ji}$ ($i \neq j$) that the coefficient of E_{ik} is $0 = \delta_{jk}1 - u_{ij}u_{ik}$, so for $k \neq i, j$ (which exists by $2n > 2$) we have $0 = (\delta_{jj}1 - u_{ij}u_{ij})u_{ik}^2 - u_{ij}(\delta_{jk}1 - u_{ij}u_{ik})u_{ik} + (\delta_{kk}1 - u_{ik}u_{ik}) = 1$, a contradiction.

$$(1.10.2) \quad \begin{aligned} \bar{1} &= 1, \quad \bar{\bar{y}} = y, \quad q(\bar{y}) = q(y), \quad q(x, \bar{y}) = q(\bar{x}, y), \\ \overline{U_x y} &= U_{\bar{x}} \bar{y}, \quad U_x \bar{x} = q(x)x. \end{aligned}$$

In what follows we will need to consider the class of **Jordan pairs of rectangular type** [9, 6.4]: if R is an associative Φ -algebra, $M = (M^+, M^-)$ a pair of Φ -modules such that M^+ is a left and M^- a right R -module, and $f : M^+ \times M^- \rightarrow R$ an R -bilinear map, then (M^+, M^-) becomes an associative pair

$$(1.11.1) \quad \begin{aligned} A(M, R, f) : \langle x^+, y^-, z^+ \rangle &:= f(x^+, y^-)z^+, \\ \langle y^-, z^+, w^- \rangle &:= y^- f(z^+, w^-), \end{aligned}$$

which naturally yields a Jordan pair

$$(1.11.2) \quad \mathcal{V}(M, R, f) := A(M, R, f)^J : \begin{aligned} Q_{x^+} y^- &= f(x^+, y^-)x^+, \\ Q_{y^-} x^+ &= y^- f(x^+, y^-). \end{aligned}$$

As is customary, we will denote **direct products** by $\prod J_i$, **subdirect products** as

$$J \cong \prod J_i \quad (J_i = \pi_i(J) \text{ for homomorphisms } \pi_i \text{ with } \bigcap \text{Ker } \pi_i = 0)$$

and the Jordan **system direct sum** $J_1 \times J_2$ as $J_1 \boxplus J_2$ (instead of the mere module direct sum $J_1 \oplus J_2$). A Jordan triple or pair comes with a slew of Jordan algebras, its **homotopes**

$$(1.12.1) \quad b\text{-homotope } J^{(b)} : \quad \begin{aligned} U_x^{(b)} y &= P_x P_b y, & x^{(2,b)} &= P_x b, \\ V_{x,y}^{(b)} &= L_{x, P(b)y}, & V_x^{(b)} &= L_{x,b}. \end{aligned}$$

To a certain extent, we can condense the bad behavior of the algebra $J^{(b)}$ into the **kernel of J at b**

$$(1.12.2) \quad \begin{aligned} \text{Ker}_J(b) &= \{z \in J \mid P_b z = P_b P_z b = 0\} \triangleleft J^{(b)}, \\ \text{Ker}_J(b) &= \{z \in J \mid P_b z = 0\} = \text{Ker}(P_b) \text{ if } J \\ &\text{is nondegenerate [7, 1.2.4],} \end{aligned}$$

so that the **local algebra of J at b** , $A_b := J^{(b)}/\text{Ker}_J(b)$, inherits a good number of desirable properties from the original system J (see [7]).

Especially important in our Clifford situation are isotopes of Jordan Clifford algebras $J(q, 1)$ as in (1.10.1):

$$(1.12.3) \quad \begin{aligned} &\text{if } q(b) \neq 0 \text{ then } J(q, 1)^{(b)} = J(q^{(b)}, 1^{(b)}) \text{ for } q^{(b)}(x) = q(b)q(x), \\ &1^{(b)} = b^{-1}; \text{ here } t^{(b)}(y) = q(y, \bar{b}), \bar{y}^{(b)} = q(y, \bar{b})b^{-1} - y \\ &= q(b)^{-1}\overline{U_b y}. \end{aligned}$$

Given an **invertible symmetric structural transformation** $J \xrightarrow{\sigma} J$ (a linear bijection such that $P_{\sigma(x)} = \sigma P_x \sigma$), we can form the **σ -isotope**

$$(1.13.1) \quad J^{(\sigma)} : \quad P_x^{(\sigma)} = P_x \sigma \quad \text{where } P_{\sigma(x)} = \sigma P_x \sigma,$$

which is again a triple (but not necessarily an algebra). Such isotopes arise naturally when comparing pairs $\mathcal{V}(J)$ built out of triples (cf. [8]):

$$(1.13.2) \quad \begin{aligned} \mathcal{V}(J_1) \simeq \mathcal{V}(J_2) \text{ via } (\varphi^+, \varphi^-) &\iff J_1 \simeq J_2^{(\sigma)} \text{ via } \varphi^+ \\ (\sigma = \varphi^- \circ (\varphi^+)^{-1}). \end{aligned}$$

Indeed, $\varphi = (\varphi^+, \varphi^-)$ is a pair isomorphism $\mathcal{V}(J_1) \longrightarrow \mathcal{V}(J_2)$ iff the φ^ε are linear bijections $J_1 \longrightarrow J_2$ with $\varphi^\varepsilon(P_{x_1}y_1) = P_{\varphi^\varepsilon(x_1)}\varphi^{-\varepsilon}(y_1)$, so $\varphi^+ : J_1 \longrightarrow J_2$ is an isomorphism of J_1 with the σ -isotope of J_2 for $\sigma = \varphi^- \circ \varphi^{+^{-1}}$. For, $\varphi^+(P_{x_1}y_1) = P_{\varphi^+(x_1)}\varphi^-(\varphi^{+^{-1}}(\varphi^+(y_1))) = P_{\varphi^+(x_1)}\sigma(\varphi^+(y_1)) = P_{\varphi^+(x_1)}\varphi^+(y_1)$ where $J_2 \xrightarrow{\sigma} J_2$ satisfies $P_{\sigma(x_2)} = \sigma \circ P_{x_2} \circ \sigma$ [write $x_2 = \varphi^+(x_1)$, $y_2 = \varphi^+(y_1)$ so $\sigma(P_{x_2}\sigma(y_2)) = \sigma(P_{\varphi^+(x_1)}\varphi^-(y_1)) = \sigma(\varphi^+(P_{x_1}y_1)) = \varphi^-(P_{x_1}y_1) = P_{\varphi^-(x_1)}\varphi^+(y_1) = P_{\sigma(x_2)}y_2$]. Conversely, if $J_1 \simeq J_2^\sigma$ via φ then $\varphi^+ = \varphi$, $\varphi^- = \sigma \circ \varphi$ is a pair isomorphism.

An important special case of (1.13.2) is (1.8.5) for even alternating A_{2n} . If $Q(\Phi) = M_2(\Phi)$ denotes the split quaternion algebra over Φ , the standard quaternion involution $\bar{a} = t(a)1 - a$ has

$$\overline{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and so takes the form $-sa^{tr_2}s = sa^{tr_2}s^{-1}$ of a conjugate by $s := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of the transpose involution tr_2 on $M_2(\Phi)$. The isomorphism $X_n \rightarrow X_{2n}$ of associative algebras $M_n(Q(\Phi)) = M_n(M_2(\Phi)) \rightarrow M_{2n}(\Phi)$ carries the *standard involution* $X_n^* := \overline{X_n}^{tr_n} = (S_n X_n^{tr_2} S_n^{-1})^{tr_n} = S_n (X_n^{tr_2})^{tr_n} S_n^{-1}$ for $S_n := \begin{pmatrix} s & \dots & 0 \\ & \dots & \\ 0 & \dots & s \end{pmatrix}$ to the *symplectic involution* $X^{*_{sp}} := S_{2n} X_{2n}^{tr_{2n}} S_{2n}^{-1}$ (a conjugate of the transpose involution tr_{2n} on $M_{2n}(\Phi)$). This induces isomorphisms

$$(1.13.3) \quad \begin{aligned} (M_n(Q(\Phi)), *) &\rightarrow (M_{2n}(\Phi), *_{sp}), \\ H_n(Q(\Phi), *) &\rightarrow H(M_{2n}(\Phi), *_{sp}), \end{aligned}$$

and followed by the isotopy $X_{2n} \rightarrow S_{2n} X_{2n}$ it then takes $H_n(Q(\Phi), *) \rightarrow Skew(M_{2n}(\Phi), tr_{2n})$ as spaces and $H_n(Q(\Phi), \Phi) \rightarrow Alt(M_{2n}(\Phi), tr_{2n}) = A_{2n}(\Phi)$ as ample subspaces, yielding the isomorphism (1.8.4). [Note that $P_{X_n}^{(\sigma)} Y_n = X_n [-S_n Y_n S_n] X_n = -X_n S_n Y_n S_n X_n$ is sent to $-S_{2n} X_{2n} \cdot S_{2n} Y_{2n} \cdot S_{2n} X_{2n} = P_{S_{2n} X_{2n}} S_{2n} Y_{2n}$. Here S_n lies in Sk_n , not H_n , so $H_n^{(\sigma)}$ is a generalized isotope as in (1.3.1).] Then (1.8.5) follows from (1.13.2).

An **inner ideal** is a subspace $K \subseteq J$ with $P_K J \subseteq K$, an **outer ideal** $B \subseteq J$ has $P_J B + L_{J,J} B \subseteq B$, and an **ideal** $I \triangleleft J$ is both inner and outer. A subspace $C \subseteq J$ is a **semi-ideal** if $P_C J + P_J P_J C + L_{J,J} C \subseteq C$.

A Jordan system is **semiprime** if it has no **trivial ideal** $P_I I = 0$, is **prime** if it has no orthogonal ideals $P_I K = 0$, is **nondegenerate** if it has no **trivial elements** $P_z = 0$, is **strongly prime** if it is both prime and nondegenerate, is **simple** if $P_J J \neq 0$ and J has no nonzero proper ideals, is a **division system** if its nonzero elements are **invertible** ($x \neq 0 \Rightarrow P_x$ is invertible), is **locally-nilpotent** if all its finitely generated subsystems are nilpotent, is **pnil (properly nil)** if all its homotopes are nil (for all $z, y \in J$ there is n , depending on z and y , such that $z^{(n,y)} = 0$), is **psnil (properly strictly nil)** if all its homotopes $J^{(y)}$ remain strictly nil, i.e. all scalar extensions are nil (for all \tilde{z} in all J_Ω and all $y \in J$ there is n such that $\tilde{z}^{(n,y)} = 0$), is **pnil- or psnil-free** if it has no nonzero pnil or psnil ideal, and is **semiprimitive** if it has no **quasi-invertible ideal** I consisting entirely of **properly quasi-invertible elements** z , where z is p.q.i. if $B_{z,y} = Id - L_{z,y} + P_z P_y$ is invertible on J for all $y \in J$.

Most undesirable properties can be removed and a desirable property created by means of radical surgery: there is a *smallest* ideal \mathcal{R} whose quotient has the desirable property (\mathcal{R} is the intersection of *all* such ideals) (see [19]). The *Jacobson radical* $\mathcal{R}ad(J)$ removes quasi-invertible ideals and creates semiprimitivity; the *properly-nil radical* $\mathcal{P}Nil(J)$ removes pnil ideals and creates pnil-freedom (analogously for the *properly-strictly-nil radical* $\mathcal{PS}Nil(J)$); the *locally nilpotent radical* $\mathcal{L}(J)$ removes locally nilpotent ideals; the *McCrimmon or lower radical* $\mathcal{M}c(J)$ removes trivial elements and creates nondegeneracy; the *Baer radical* $\mathcal{B}(J)$ removes nilpotent ideals and creates semiprimeness (see [20]). In general, $\mathcal{M}c(J)$ need not consist entirely of trivial elements (Zel'manov showed it consists precisely of the m -finite elements [28, 21, Th.2.6 p.155], and $\mathcal{B}(J)$ need not itself be nilpotent, but $\mathcal{R}ad(J)$ is the maximal q.i. ideal, $\mathcal{P}Nil(J)$ is the maximal pnil ideal (analogously for $\mathcal{PS}Nil(J)$), and $\mathcal{L}(J)$ is the maximal locally nilpotent ideal. Recall that

$$(1.14.1) \quad \mathcal{R}ad(J) \supseteq \mathcal{P}Nil(J) \supseteq \mathcal{PS}Nil(J) \stackrel{(*)}{\supseteq} \mathcal{L}(J) \stackrel{(**)}{\supseteq} \mathcal{M}c(J) \supseteq \mathcal{B}(J),$$

in general, and all coincide in the finite situation [13, 10.8, 14.8, 14.10, 14.11]:

$$(1.14.2) \quad \mathcal{R}ad(J) = \mathcal{P}Nil(J) = \mathcal{PS}Nil(J) = \mathcal{L}(J) = \mathcal{M}c(J) = \mathcal{B}(J)$$

if J is finite-dimensional over a field.

The inclusion $(**)$ emerges as a byproduct of Kostrikin and Zel'manov's work on Lie sandwiches (as a consequence, a simple triple or pair is necessarily nondegenerate) and $(*)$ holds since locally-nilpotent ideals remain so in scalar extensions and $\mathcal{L}(J)$ is not only nil but pnil in J : given $z \in \mathcal{L}(J)$, $w \in J$, we have $\langle z, P_w z \rangle \subseteq \mathcal{L}(J) \Rightarrow \langle z, P_w z \rangle$ nilpotent of some index n , so all products of degree $\geq n$ vanish, $0 = z^{(n, P_w z)} = z^{(2n-1, w)}$, so z is nilpotent in $J^{(w)}$. Let us also mention that if $\tilde{J} \supseteq J$ is a **tight cover** of J ,

$$(1.15) \quad \text{all ideals of } \tilde{J} \text{ hit } J : \quad 0 \neq \tilde{I} \triangleleft \tilde{J} \implies \tilde{I} \cap J \neq 0,$$

then \tilde{J} behaves like J in many ways,

$$(1.16) \quad \text{if } J \text{ is semiprime, prime, nondegenerate, pnil-free or psnil-free, so is any tight cover } \tilde{J}.$$

We say a Jordan pair $V = (V^+, V^-)$ is ε -**primitive** at $b^{-\varepsilon}$ if it has a $b^{-\varepsilon}$ -**primitizer**, that is, a proper inner ideal $K^\varepsilon < V^\varepsilon$ modular in the

homotope $V^{\varepsilon(b^{-\varepsilon})}$ with some modulus b^ε , which complements nonzero pair ideals:

$$(1.17.1) \quad B_{b^\varepsilon, b^{-\varepsilon}} V^\varepsilon \subseteq K^\varepsilon$$

$$(1.17.2) \quad b^\varepsilon - Q_{b^\varepsilon} b^{-\varepsilon} \in K^\varepsilon$$

$$(1.17.3) \quad D_{b^\varepsilon, b^{-\varepsilon}} K^\varepsilon \subseteq K^\varepsilon$$

$$(1.17.4) \quad \{D_{x^\varepsilon, b^{-\varepsilon}} - D_{b^\varepsilon, Q(b^{-\varepsilon})x^\varepsilon}\} K^\varepsilon \subseteq K^\varepsilon \quad \text{for all } x^\varepsilon \in V^\varepsilon$$

$$(1.17.5) \quad 0 \neq I \triangleleft V \implies I^\varepsilon + K^\varepsilon = V^\varepsilon.$$

[Note that this latter implies $I \neq 0 \implies I^\varepsilon \neq 0$, equivalently that V is ε -**core-free**, $Q_{V^\varepsilon}(z^{-\varepsilon}) = Q_{V^\varepsilon} Q_{z^{-\varepsilon}} V^\varepsilon = 0 \implies z^{-\varepsilon} = 0$]. V is **primitive** if it is ε -primitive for some ε . (Note that the superscript-less version of (1.17.1-5) provides the definition of primitivity for triples.) Primitivity of V and $Pol(V)$ are equivalent [7, 5.5]:

$$(1.18) \quad V = (V^+, V^-) \text{ is a primitive pair} \\ \iff Pol(V) = V^+ \oplus V^- \text{ is a primitive triple system.}$$

Finally, polynomial identities will play a crucial role in our classification. A Jordan system (triple or pair) is **i-special** if it satisfies all the identities satisfied by all special Jordan systems (the so-called **s-identities**). An s-identity is just a Jordan polynomial f whose image f^s in the free special Jordan system is zero. To avoid speciality and torsion issues for scalars from a general ring Φ , one always restricts the term *polynomial identity* to polynomials which are *s-monic*, where f is **s-monic** if its image f^s is not merely nonzero, but is **monic** (has a leading coefficient 1, i.e. one of its associative monomials of maximal total x -degree appears with coefficient 1). A **homotope PI** (or **HPI**) is an s-monic Jordan polynomial $f^{(y)}(x_1, \dots, x_n)$ in the y -homotope of the free Jordan system. We say that a Jordan system J **satisfies** the HPI if

$$(1.19) \quad f^{(y)}(x_1, \dots, x_n) \text{ vanishes in all scalar extensions} \\ J_\Omega \text{ of } J, \text{ i.e. } f^{(b)}(a_1, \dots, a_n) = 0 \text{ for all } a_i, b \in J_\Omega$$

(hence all homotopes $J^{(b)}$ satisfy an ordinary PI). Note that for us, “satisfies” always means “strictly satisfies”.

A Jordan system is of **Clifford type** if it satisfies a **Clifford HPI**, i.e. an s -monic polynomial that does not vanish on the split matrix algebra $H_3 = H(M_3(\Phi), t)$ of hermitian 3×3 matrices over Φ ; in particular, by definition (1.19), such a polynomial is not satisfied by any system J having a scalar extension ΩJ containing H_3 .

In the presence of any HPI (not necessarily Clifford), a primitive system possesses a nonzero socle, a fact that we record for future reference (this is the main result of [7], Theorems 6.1 / 6.2 pp. 232 / 233; also see [14] for generalities on socles):

$$(1.20) \quad \begin{array}{l} \text{if a primitive triple (or pair) } J \text{ satisfies an HPI,} \\ \text{then } \text{Soc}(J) \neq 0. \end{array}$$

We will see that if the identity is actually Clifford, then J has finite capacity ≤ 2 .

2. Classical pairs of Clifford type

To establish a list of candidates for our classification, we use Loos' list [13, p.138] of classical central simple Jordan pairs as a starting point. Loos' list was based on Jacobson's list of simple Jordan algebras of finite capacity, which included Type (0), division algebras; from the later work of Zel'manov we know these are all subsumed under the remaining types. Loos' list did not specify the centroid. Jordan centroids were examined in detail in [22, Theorem 5.9] specifically to culminate in:

Loos' Classical List 2.1. If V is a central simple Jordan pair of finite capacity over the field Φ , then V is isomorphic to one of the following:

- (I) *Rectangular Pairs*: $\mathcal{V}(M, R, f)$ for a central simple Artinian Φ -algebra $R \simeq M_n(\Delta)$ for some central division algebra Δ over Φ , f a nondegenerate R -bilinear pairing,
- (II) *Alternating Pairs*: $\mathcal{V}(A_n(\Phi))$ for $n \geq 5$,
- (III) *Clifford Pairs*: $\mathcal{V}(J)$ for a central-simple Clifford Jordan algebra J (a tight scalar form of some simple $J(\bar{q}, 1)$ for a nondegenerate quadratic form \bar{q} of dimension $\neq 2$ over an algebraically closed extension field $\bar{\Phi}$ of Φ),
- (IV) *Hermitian Pairs*: $\mathcal{V}(H_n(\Delta, \Delta_\circ))$ for $n \geq 2$ and some central division

- algebra Δ with involution over Ω and ample subspace Δ_\circ , $\Phi = C(\Delta, \Delta_\circ)$
 $= \{\alpha \in C(\Delta) \mid \alpha\Delta_\circ \subset \Delta_\circ\}$,
- (V) *BiCayley Pairs*: $\mathcal{V}(M_{1,2}(\mathcal{C}))$ for some 8-dimensional Cayley algebra \mathcal{C} over Φ ,
 - (VI) *Albert Pairs*: $\mathcal{V}(\mathcal{A})$ for some 27-dimensional exceptional simple Albert algebra \mathcal{A} over Φ .

Since we are investigating i-special systems, we immediately eliminate the BiCayley and Albert pairs (since they are i-exceptional by [18, 3.11]); this leaves us with types (I)-(IV). Next, we remark that many small systems can be subsumed under the Clifford case III: $\mathcal{V}(H_2(\Delta, \Phi))$ is Clifford, as is $\mathcal{V}(H_1(\Delta, \Delta_0))$ if Δ is a composition algebra over Φ . Similarly, $\mathcal{V}(A_n(\Phi))$ is Clifford for $n \leq 4$ since $\mathcal{V}(A_4(\Phi)) \simeq \mathcal{V}(H_2(Q(\Phi), \Phi))$ by (1.8.5).

We now determine which of the remaining possibilities can satisfy a Clifford HPI. In this case no homotope of the pair V can contain a copy of $\mathcal{V}(H_3(\Phi))$, and since any scalar extension V_Ω continues to satisfy the Clifford HPI, V_Ω can't contain $\mathcal{V}(H_3(\Omega))$ either. This rules out all alternating $\mathcal{V}(A_n(\Phi))$ for $n \geq 6$ (they contain $\mathcal{V}(A_6(\Phi)) \simeq \mathcal{V}(H_3(Q(\Phi), \Phi))$) and in view of the above comment for $n \leq 4$, only $\mathcal{V}(A_5(\Phi))$ remains. Next, we claim the hermitian pairs can be eliminated: they are either “too big” or already Clifford. The following general theorem explains why and also provides a useful argument for division and rectangular pairs.

Theorem 2.2 (Hermitian-Clifford). *If R is a $*$ -simple Artinian algebra over its $*$ -center Φ_\circ and $J = H_\circ(R, *)$ an ample Φ_\circ -subspace of symmetric elements which satisfies a Clifford HPI, then J is a Jordan Clifford algebra $J(q, 1)$ for a nondegenerate quadratic form q of dimension 1, 3, 4 or 6 over Φ_\circ . In particular (applying this to $R \oplus R^{op}$ under the exchange involution, $\Phi_\circ = \Phi$), if R is a simple Artinian algebra over its center Φ then R^+ satisfies a Clifford HPI if and only if either $R = \Phi 1$ is 1-dimensional or $R = \mathcal{Q}$ is a 4-dimensional quaternion algebra over Φ .*

PROOF. From associative theory we recall some facts about splitting fields (c.f. [11, Th.3 p.120]). Any central simple Artinian algebra is $R \simeq M_n(\Delta)$ over a division algebra Δ with center Φ and degree r (finite or ∞), so some Δ_Ω contains $M_m(\Omega)$ for all finite $m \leq r$. Then R has degree $s = nr$ so that some scalar extension $R_\Omega = \Omega \otimes_\Phi R$ contains $M_m(\Omega)$ for all finite $m \leq s$. We denote this situation by $R \approx M_s(\Omega)$. If $J \supset \text{trace}(R, *)$ satisfies

a Jordan homotope PI then R satisfies an associative $*$ -PI, hence by Amitsur [1] an ordinary PI. Once R strictly satisfies some polynomial identity, then $s < \infty$ and R is finite-dimensional over its center (of dimension s^2). When $s \leq 2$, this says $R = \Phi 1$ of dimension 1 ($s = nr = 1 \implies n = r = 1$) or $R = \mathcal{Q}$ of dimension 4 ($s = 2$ implies either $n = 1, r = 2$ and $\mathcal{Q} = \Delta$ a quaternion division algebra, or $n = 2, r = 1$ and $\mathcal{Q} = M_2(\Phi)$ split quaternions):

$$(1) \quad s = 1 \implies R = \Phi 1, \quad s = 2 \implies R = \mathcal{Q} \text{ quaternions.}$$

Next we bring in the involution. If $(R, *)$ is $*$ -simple over its $*$ -center Φ_\circ , any involution is of **exchange**, **orthogonal** or **symplectic type** (c.f. [12, Sec.5.1 pp. 186-193], [8, Th.1.2.2 p.17]).

Now $*$ has **exchange type** iff it is an “involution of second kind”, where its center $\Phi > \Phi_\circ$ is bigger than its $*$ -center (i.e. $*$ doesn't fix the center), in which case $[\Phi : \Phi_\circ] = 2$. If R is not simple then $\Phi = \Phi_\circ \boxplus \Phi_\circ$, and $*$ is already an exchange involution on $R = T \boxplus T^* \simeq T \boxplus T^{op}$ for central simple T over Φ_\circ , with $H(R, *) \simeq T^+$. For a splitting extension $\Omega \supseteq \Phi_\circ$ for T we have $(R, *)_\Omega \simeq (M_s(\Omega) \boxplus M_s(\Omega)^{op}, ex)$, $H(R, *)_\Omega \simeq M_s(\Omega)^+$. If R is simple (e.g. $R = M_n(C), * = \text{conjugate transpose}, \Phi = C, \Phi_\circ = R$), then Φ is a field, and $(R, *) \otimes_{\Phi_\circ} \Phi \simeq (R \boxplus R^{op}, ex)$ splits into the form of an exchange involution, so for a splitting extension $\Omega \supseteq \Phi \supseteq \Phi_\circ$ for R we have $(R, *)_\Omega \simeq ((R, *)_\Phi)_\Omega \simeq (R \boxplus R^{op}, ex)_\Omega = (R_\Omega \boxplus R_\Omega^{op}, ex) \simeq (M_s(\Omega) \boxplus M_s(\Omega)^{op}, ex)$, hence again

$$H(R, *)_\Omega \simeq M_s(\Omega)^+.$$

Here the only ample subspace under the exchange involution (where all symmetric elements are traces) is the full $H(R, *)_\Omega$, and this satisfies a Clifford HPI iff $s \leq 2$: if $s \geq 3$ then $M_s(\Omega) \supset H_3(\Phi)$, thus $s = 1$ or 2 and by (1) $R = \Phi$ or \mathcal{Q} , so

$$(2) \quad \begin{aligned} &\text{if } J = H_\circ(R, *) \text{ of exchange type satisfies a Clifford HPI} \\ &\text{then } R = \Phi \text{ or } R = \mathcal{Q} \text{ for a quaternion algebra } \mathcal{Q}, J = \Phi_\circ \\ &\text{or } H_\circ(\mathcal{Q}, *) \text{, so } J = J(q, 1) \text{ for } q \text{ of dimension 1 or 4 over } \Phi_\circ. \end{aligned}$$

Now consider $*$ of **orthogonal type** (of “first kind”, so center = $*$ -center and $\Phi = \Phi_\circ$) where $(R, *)_\Omega \simeq (M_s(\Omega), t)$ i.e. $*$ becomes the transpose involution,

$$H(R, *)_\Omega \simeq H_s(\Omega, t),$$

whose only ample Ω -subspace is $H_s(\Omega, t)$ itself. Again if $H_o(R, *)$ (hence $\Omega J = H_s(\Omega, t)$) satisfies a Clifford HPI then $s \leq 2$ since if $s \geq 3$ the ample $H_5(\Omega, t)$ would contain $H_3(\Phi, t)$. Once more by (1), $R = \Phi$ or \mathcal{Q} and $H(R, *) = \Phi 1$ or $H(\mathcal{Q}, *)$, and the ample Φ -subspace J is all of $H(R, *)$ of dimension 1 or 3,

- if $J = H_o(R, *)$ of orthogonal type satisfies a Clifford HPI
- (3) then $R = \Phi$ or \mathcal{Q} , $J = \Phi$ or $J = H(\mathcal{Q}, *)$ for a quaternion algebra \mathcal{Q} , so $J = J(q, 1)$ for q of dimension 1 or 3 over Φ .

Finally, consider $*$ of **symplectic type** where $(R, *)_\Omega \simeq (M_s(\Omega), sp) = (M_{2r}(\Omega), sp)$. Here $s = 2r$ must be even. We saw in (1.13.3) that the symplectic involution is isomorphic to the standard conjugate transpose $X^* = \overline{X}^{trr}$ on $M_r(Q(\Omega))$, so

$$H(R, *)_\Omega \simeq H(M_{2r}(\Omega), *_{sp}) \simeq H_r(Q(\Omega), *).$$

Here the Ω -ample subspace ΩJ contains $H_r(\Omega)$ (any ample Ω -subspace has at least Ω down its diagonal), so again a Clifford HPI forces $r \leq 2$ (this time $s \leq 4$). If $r = 1$, $s = 2$, then $R = \mathcal{Q}$, $H(R, *) = H(\mathcal{Q}, *)$ and $J_\Omega = H_o(R, *) \otimes_\Phi \Omega$ is an ample subspace of $H(Q(\Omega), *)$. In characteristic not 2, $H(Q(\Omega), *) = \Omega 1$ has dimension 1 over Ω , hence J has dimension 1 over Φ . In characteristic 2, $H(Q(\Omega), *)$ consists of all trace zero quaternions, hence has dimension 3, and only admits ample subspaces of dimensions 1 or 3 over Ω , hence J has dimension 1 or 3 over Φ . In all cases $J = \mathcal{J}(q, 1)$ of dimension 1 or 3 over Φ .

Now we come to the hard part, the case $r = 2$, $s = 4$. In characteristic $\neq 2$ the only ample subspace of $H(Q(\Omega), -) = \Omega 1$ is itself, and the only ample subspace of $H_2(Q(\Omega), *)$ is $H_2(Q(\Omega), \Omega, *) = \{X = \begin{pmatrix} \alpha & a \\ \bar{a} & \beta \end{pmatrix} \mid \alpha, \beta \in \Omega, a \in \mathcal{Q}\}$ of dimension $1 + 1 + 4 = 6$; this is $J(det, 1)$ for the quadratic form $det(X) = \alpha\beta - a\bar{a} \in \Omega 1$. Then $J = H_2(Q, \Phi, *)$ is also $J(q, 1)$ of dimension 6 for q the restriction of det .

In characteristic 2, in addition to the 1-dimensional ample subspace $\Omega 1$ of $H(Q(\Omega), -)$ as above, there is the entire 3-dimensional space $H(Q(\Omega), -) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Q(\Omega) \right\}$; the corresponding ample subspace of $H_2(Q(\Omega), *)$ is

$H_2(\mathcal{Q}(\Omega), *) = \{Y = \begin{pmatrix} h & a \\ \bar{a} & k \end{pmatrix} \mid a \in \mathcal{Q}, h, k \in H(\mathcal{Q}(\Omega), -)\}$ of dimension $3 + 3 + 4 = 10$. Here there are only two orthogonal idempotents, yet the algebra is NOT an algebra $J(q, 1)$! But the HPI hypothesis comes to our aid: all homotopes $(J_\Omega)^{(b)}$ must still satisfy the Clifford PI, and

$$(4) \quad (J_\Omega)^{(b)} \simeq H(M_4(\Omega), tr_4) \supset H_4(\Phi)$$

$$(b := B_2 := \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \in H_2(\mathcal{Q}(\Omega), *), u := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in H(\mathcal{Q}(\Omega), -))$$

has 4 orthogonal idempotents and does NOT satisfy the Clifford PI. Indeed, IN CHARACTERISTIC 2 B_2 COINCIDES WITH $S_2 := \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$ ($s := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in Sk(\mathcal{Q}(\Omega), -)$), so in $M_4(\Omega)$ we have

$$B_4 = S_4 = -S_4 = S_4^{-1} \quad (\text{char.} = 2).$$

From this we see the B_4 -conjugate of the symplectic involution on $M_4(\Omega)$ is just the trace involution, $X^{*_{sp}(B_4)} := B_4 X^{*_{sp}} B_4^{-1} = B_4 (S_4 X^{tr_4} S_4^{-1}) B_4^{-1} = X^{tr_4}$, hence taking isotopes in (1.13.3) $J_\Omega = H_2(\mathcal{Q}(\Omega), *) \simeq H(M_4(\Omega), *_{sp})$ yields $J_\Omega^{(b)} \simeq H(M_4(\Omega)^{(B_4)}, *_{sp}) \simeq H(M_4(\Omega), *_{sp}(B_4))$ (taking the isotope of the involution in place of the algebra) $= H(M_4(\Omega), tr_4)$, as claimed in (4).

Let $J = H_\circ(R, *)$ of symplectic type satisfy a Clifford HPI.

If $s = 2$ then $R = \mathcal{Q}$, $J = \Phi$ of dimension 1, or $H_\circ(\mathcal{Q}, *)$ of

- (5) dimension 3 (only in characteristic 2), so $J = J(q, 1)$ for q of dimension 1 or 3. If $s = 4$ then $R = M_2(\mathcal{Q})$, $J = H_2(\mathcal{Q}, \Phi, *_{s}) = J(q, 1)$ for q of dimension 6 over Φ . ■

Thus applying the above theorem to the case $R = M_n(\Delta)$, all the hermitian pairs (type IV) satisfying a Clifford HPI are Clifford pairs (type III). Note that any simple $J(q, 1)$ over a field Φ is a tight scalar form of a simple $J(\bar{q}, \bar{1})$ over $\bar{\Phi}$ by the Clifford Tightening Theorem [23, Thm. 3.4], and that over an algebraically closed field $J(\bar{q}, \bar{1})$ is simple iff \bar{q} is nondegenerate of dimension $\neq 2$.

Remark 2.3. In getting rid of division pairs from Loos' list, we could avoid the high-powered tool of the Zel'manov classification by using big field extensions. If J is a division algebra over its centroid Φ and we form a tight scalar extension $J \subset \tilde{J} = \tilde{\Phi}J$ for $\tilde{\Phi} \supset \Phi$ a big algebraically closed field, then by tightness \tilde{J} remains simple but is no longer a division algebra unless we have the trivial case $\tilde{J} = \tilde{\Phi}1$ (in which case $J = \tilde{\Phi}_\circ 1$ is tiny Clifford): if $\tilde{x} \notin \tilde{\Phi}1$ then all $\tilde{\omega}1 - \tilde{x}$ are nonzero, $\tilde{\Phi} \subset \text{Resolvent}(\tilde{x})$ and by Amitsur's Resolvent Theorem (see Theorem 8.5 below) and bigness, \tilde{x} is algebraic over $\tilde{\Phi}$, hence by algebraic closure $p(\tilde{x}) = \prod(\tilde{x} - \tilde{\alpha}_i 1) = 0 \Rightarrow$ some $\tilde{x} - \tilde{\alpha}_i 1 = 0$ and $\tilde{x} = \tilde{\alpha}_i 1 \in \tilde{\Phi}1$ after all. ■

Thus, by imposing a Clifford HPI on the classical pairs of 2.1, we have shrunk our list of Cliffordian candidates to

List 2.4. If V is an i -special central simple Jordan pair of finite capacity over the field Φ satisfying a Clifford HPI, then V is isomorphic to one of the following:

- (I) *Rectangular Pairs:* $\mathcal{V}(M, R, f)$ for a central simple Artinian Φ -algebra $R \simeq M_n(\Delta)$ (Δ a central division algebra over Φ) and a nondegenerate bilinear pairing f on M ,
- (II) *Alternating Pair:* $\mathcal{V}(A_5(\Phi))$,
- (III) *Clifford Pairs:* $\mathcal{V}(J)$ for a central-simple Clifford Jordan algebra J (a tight scalar form of some simple $J(\bar{q}, 1)$ for a nondegenerate quadratic form \bar{q} of dimension $\neq 2$ over an algebraically closed extension field $\bar{\Phi}$ of Φ).

3. Rectangular Pairs

Seeking a more explicit description of rectangular pairs, we now probe into the structure of $\mathcal{V}(M, R, f)$ when R is a simple Artinian associative algebra: it can be realized as a (potentially very large) pair of rectangular matrices $\mathcal{M}_{p,q}(\Delta) := (M_{p,q}(\Delta), M_{q,p}(\Delta))$ where we may always assume $p \leq q$. Since a Clifford HPI puts a limit on the number of pair idempotents, a rectangular pair of Clifford type must have $p \leq 2$ (while q may actually be infinite). We begin with

Proposition 3.1. *If $V = \mathcal{V}(M, R, f)$ for a simple Artinian algebra $R \simeq M_n(\Delta)$ (Δ a division algebra over Φ) and a nondegenerate bilinear pairing f on M , then either*

$$(i) \ V \simeq \mathcal{M}_{n,q}(\Delta) \text{ for some } q,$$

or

$$(ii) \ V \supset \mathcal{M}_{n,q}(\Delta) \text{ for all } q.$$

When $V \supset \mathcal{M}_{n,q}(\Delta)$, there exists $b^- \in V^-$ with $\text{rank}(b^-) = \min(n, q)$.

PROOF. Abstractly we know that since M^+ is a unital left R -module, it is a direct sum of irreducibles isomorphic to $Re_{11} \simeq M_{n,1}(\Delta)$: $M^+ = \bigoplus_{i \in I} M_i^+ \simeq \bigoplus_{i \in I} M_{n,1}(\Delta) =: M_{n,I}(\Delta)$, and dually $M^- \simeq M_{J,n}(\Delta)$. However, this module decomposition completely disregards f . We must use a straightening argument to construct an “ f -orthonormal basis” $\{m_k^\pm\}$:

$$(1) \quad f(m_i^+, m_j^-) = \delta_{ij} e_{11} \quad (m_i^+ = e_{11} m_i^+, m_j^- = m_j^- e_{11}).$$

We do this by induction. We start with no basis vectors at all: $q - 1 = 0$ for $q = 1$. Suppose we have constructed f -orthonormal vectors m_k^\pm for $k = 1, 2, \dots, q - 1$ as in (1). We claim we can split off the R -span of these vectors:

$$(2) \quad \begin{aligned} M^+ &= Rm_1^+ \oplus \dots \oplus Rm_{q-1}^+ \oplus M_q^+, \text{ where } Rm_i^+ \simeq M_{n,1}(\Delta) \\ &\text{and } M_q^+ = \{x^+ \mid f(x^+, m_j^-) = 0 \text{ for } j = 1, \dots, q - 1\}, \\ M^- &= m_1^- R \oplus \dots \oplus m_{q-1}^- R \oplus M_q^-, \text{ where } m_i^- R \simeq M_{1,n}(\Delta) \\ &\text{and } M_q^- = \{x^- \mid f(m_i^+, x^-) = 0 \text{ for } i = 1, \dots, q - 1\}. \end{aligned}$$

Indeed, any m^+ can be written as $m^+ = \sum_{i=1}^{q-1} a_i m_i^+ + n^+$ ($a_i := f(m^+, m_i^-)$ $= f(m^+, m_i^- e_{11}) = a_i e_{11}$) where

$$\begin{aligned} f(n^+, m_j^-) &= f(m^+ - \sum_{i=1}^{q-1} a_i m_i^+, m_j^-) \\ &= f(m^+, m_j^-) - \sum_{i=1}^{q-1} a_i f(m_i^+, m_j^-) \\ &= a_j - \sum_{i=1}^{q-1} a_i \delta_{ij} e_{11} \text{ (by (1))} \\ &= a_j - a_j e_{11} = 0. \end{aligned}$$

Thus $n^+ \in M_q^+$, and $M^+ = \sum_{i=1}^{q-1} Rm_i^+ + M_q^+$ is a sum of modules. The sum is direct since $\sum a_i m_i^+ + n^+ = 0 \Rightarrow 0 = f(\sum a_i m_i^+ + n^+, m_j^-) = a_j e_{11} = a_j$ for $j = 1, \dots, q-1$, hence $n^+ = 0$ too. We have $Rm_i^+ \simeq Re_{11}$ since $\varphi_i^+(rm_i^+) := f(rm_i^+, m_i^-) = re_{11}$ and $\psi_i^+(re_{11}) := re_{11}m_i^+ = rm_i^+$ are well-defined inverse R -linear maps. Dually for M^- .

If $M_q^+ = M_q^- = 0$ then we are done: $M^+ \simeq \bigoplus_{i=1}^{q-1} M_{n,1}(\Delta) = M_{n,q-1}(\Delta)$, $M^- \simeq M_{q-1,n}(\Delta)$ where $f(m^+, n^-) = f(\sum_i a_i m_i^+, \sum_j m_j^- b_j) = \sum_{i,j} a_i f(m_i^+, m_j^-) b_j = \sum_j a_j e_{11} b_j$, which in matrix terms corresponds to the product $\varphi^+(m^+) \varphi^-(n^-) = [r_{ij}] [s_{kl}]$ for $r_{ij}, s_{kl} \in \Delta$ defined by $r_{ij} e_{i1} = e_{ii} \varphi_j^+(a_j m_j^+) = \varphi_j^+(e_{ii} a_j m_j^+) = e_{ii} a_j e_{11}$, $s_{jk} e_{1k} = \varphi_j^-(m_j^- b_j) e_{kk} = e_{11} b_j e_{kk}$ [the ik -entry of the matrix product is given by

$$\begin{aligned} \left(\sum_j r_{ij} s_{jk} \right) e_{ik} &= \sum_j (r_{ij} e_{i1}) (s_{jk} e_{1k}) = \sum_j e_{ii} a_j e_{11} \cdot e_{11} b_j e_{kk} \\ &= \sum_j e_{ii} a_j e_{11} b_j e_{kk} = e_{ii} \left(\sum_j a_j e_{11} b_j \right) e_{kk} \end{aligned}$$

which equals $e_{ii}(f(m^+, n^-))e_{kk}$. Therefore we obtain the isomorphism $V \simeq \mathcal{V}((M_{n,q-1}(\Delta), M_{q-1,n}(\Delta)), R, \pi)$ under $\pi(A, B) = AB$.

If one of M_q^\pm is nonzero so is the other since they are nondegenerately paired: we have $f(n^+, M_q^-) = 0 \Rightarrow f(n^+, M^-) = 0$ ($n^+ \perp$ all $m_j^- R$) $\Rightarrow n^+ = 0$ by left nondegeneracy of f , and dually on the other side. Thus $f(M_q^+, M_q^-)$ is a nonzero ideal of the simple algebra R , so it must be all of R . In particular, $e_{11}f(M_q^+, M_q^-) \neq 0$ so $f(e_{11}M_q^+, M_q^-) \neq 0$, and there exists nonzero $n^+ \in e_{11}M_q^+$ (hence $n^+ = e_{11}n^+$) with $f(n^+, M_q^-) \neq 0$. For any such we can complete it to form a larger orthonormal basis: $0 \neq f(n^+, M_q^-) = e_{11}f(n^+, M_q^-) \subset e_{11}R \Rightarrow f(n^+, M_q^-) = e_{11}R$ (by minimality of the right ideal $e_{11}R$) $\Rightarrow f(n^+, n^-) = e_{11}$ for $n^+ = e_{11}n^+$, $n^- = n^- e_{11}$; since these are in M_q^\pm orthogonal to the m_i^\mp for $i = 1, \dots, q-1$, taking $n_q^+ = n^+$, $n_q^- = n^-$ we obtain an f -orthonormal set of length q , as in (1).

Thus either this process comes to a halt at some q , and we have $V \simeq \mathcal{M}_{n,q}(\Delta)$, or else $V \supset \mathcal{M}_{n,q}(\Delta)$ for all q .

Note that if $r = \min(n, q)$ then $b^- = \sum_{i=1}^r E_{ii}$ lies in $M_{n,q}(\Delta) \subset V^-$, $c^+ = \sum_{i=1}^r E_{ii}$ lies in $M_{q,n}(\Delta) \subset V^+$, and $f(c^+, b^-) = \sum_{i=1}^r E_{ii}$ has rank $r = \text{rank}(b^-)$. \blacksquare

In the above situation, if $q = \infty$ (i.e. $V \supset \mathcal{M}_{n,q}(\Delta)$ for all q) then clearly $r = \min(n, q) = n$, but if $q < n$ then the local algebras $A_{b^-} \simeq M_r(\Delta)$

for $r \leq q$ always comes out smaller than $R = M_n(\Delta)$! This just means that we described V wrong: instead of $V \simeq \mathcal{M}_{n,q}(\Delta) = \mathcal{V}((M_{n,q}(\Delta), M_{q,n}(\Delta)), M_n(\Delta), \pi)$, we should have described it as $V \simeq \mathcal{M}_{q,n}(\Delta) = \mathcal{V}((M_{q,n}(\Delta), M_{n,q}(\Delta)), M_q(\Delta), \pi)$.

Renormalization Lemma 3.2. *In the case of a rectangular pair $V \simeq \mathcal{M}_{p,q}(D) = (M_{p,q}(D), M_{q,p}(D))$ for an associative algebra D and finite p, q , we may always assume $p \leq q$: indeed, $\mathcal{M}_{p,q}(D) \simeq \mathcal{M}_{q,p}(D^{op})$ via $x \longrightarrow x^t$.*

PROOF. The transpose sets up linear bijections $\varphi^+ : M_{p,q}(D) \longrightarrow M_{q,p}(D^{op})$, $\varphi^- : M_{q,p}(D) \longrightarrow M_{p,q}(D^{op})$, and these yield a pair isomorphism since they preserve products: for $x \in M_{p,q}(D)$, $y \in M_{q,p}(D)$ we have $\varphi^+(Q_x y) = (x y x)^t = x^t y^t x^t = \varphi^+(x) \varphi^-(y) \varphi^+(x)$ and $\varphi^-(Q_y x) = (y x y)^t = y^t x^t y^t = \varphi^-(y) \varphi^+(x) \varphi^-(y)$, recalling $(u \cdot_D v)^t = v^t \cdot_{D^{op}} u^t$ for $u \in M_{r,s}(D), v \in M_{s,t}(D)$ [their ij -entries are $(u \cdot_D v)_{ji} = \sum_{k=1}^s u_{jk} v_{ki}$ and $(v^t \cdot_{D^{op}} u^t)_{ij} = \sum_{k=1}^s (v^t)_{ik} \cdot_{D^{op}} (u^t)_{kj} = \sum_{k=1}^s u_{jk} \cdot_D v_{ki}$ respectively]. ■

Thus we always choose to represent $\mathcal{V}(M, R, f)$ as $\mathcal{M}_{p,q}(\Delta)$ for $p \leq q$ (allowing q to be “infinity”). We say $\mathcal{V}(M, R, f)$ is of **type** $\mathcal{M}_{p,q}(\Delta)$ if it is $\simeq \mathcal{M}_{p,q}(\Delta)$ for finite $p \leq q < \infty$, and of **type** $\mathcal{M}_{p,\infty}(\Delta)$ if it contains copies of $\mathcal{M}_{p,q}(\Delta)$ for all finite q (then abstractly $M^+ \simeq M_{p,I}(\Delta)$, $M^- \simeq M_{J,p}(\Delta)$ where both index sets I, J are infinite). Combining our matrix description with a Clifford HPI, we obtain

Theorem 3.3 (Rectangular-Clifford). *If $V = \mathcal{V}(M, R, f)$ for a central simple Artinian algebra R over Φ and nondegenerate bilinear pairing f satisfies a Clifford HPI, then V has type $\mathcal{M}_{1,q}(\Phi)$ for $q \geq 1$, $\mathcal{M}_{1,q}(\mathcal{Q})$ for $q \geq 1$ and \mathcal{Q} a quaternion division algebra over Φ , or $\mathcal{M}_{2,q}(\Phi)$ for $q \geq 2$. Moreover, the pairs $\mathcal{M}_{1,1}(\Phi)$, $\mathcal{M}_{1,1}(\mathcal{Q})$ and $\mathcal{M}_{2,2}(\Phi)$ have invertible elements and are all isomorphic to a Clifford pair of type III for some nondegenerate quadratic form.*

PROOF. The last statement is clear since $\mathcal{M}_{r,r}(D) = (M_r(D), M_r(D)) \simeq \mathcal{V}(M_r(D)^+)$, and $M_1(\Phi)^+ \simeq \Phi$, $M_1(\mathcal{Q})^+ \simeq \mathcal{Q}^+ = J(q, 1)$ ($q(X) = \det(X)$), $M_2(\Phi)^+ = J(q, 1)$ (again for $q(X) = \det(X)$).

The first statement is again a consequence of Theorem 2.2. We know V is of type $\mathcal{M}_{n,q}(\Delta)$ for some $q \geq n$, in particular $V \supset W \simeq \mathcal{M}_{n,n}(\Delta) = \mathcal{V}(M_n(\Delta)^+)$. We now apply 2.2 to $R = M_n(\Delta)$ and deduce that $R = \Phi 1$ or $R = \mathcal{Q}$ quaternions over Φ ; consequently, either $n = 1$ and $\Delta = \Phi$ or $\Delta = \mathcal{Q}$

quaternions over Φ (in which case $V \simeq \mathcal{M}_{1,q}(\Phi)$ or $\mathcal{M}_{1,q}(\mathcal{Q})$, $1 \leq q \leq \infty$) or $n = 2$ and $\Delta = \Phi$ (in which case $V \simeq \mathcal{M}_{2,q}(\Phi)$, $2 \leq q \leq \infty$). \blacksquare

In view of Theorem 3.3, we can now refine List 2.4:

Clifford List 3.4. If V is an i -special central simple Jordan pair of finite capacity over a field Φ satisfying a Clifford HPI, then V is isomorphic to one of the following:

- (I) *Rectangular Pairs:* $\mathcal{V}(M, R, f)$ for a central simple Artinian Φ -algebra R of degree ≤ 2 , which thus has one of the three forms
 - (Ia) $\mathcal{M}_{1,q}(\Phi)$, $1 \leq q \leq \infty$,
 - (Ib) $\mathcal{M}_{1,q}(\mathcal{Q})$, \mathcal{Q} a quaternion division algebra over Φ , $1 \leq q \leq \infty$,
 - (Ic) $\mathcal{M}_{2,q}(\Phi)$, $2 \leq q \leq \infty$,
- (II) *Alternating Pair:* $\mathcal{V}(A_5(\Phi))$,
- (III) *Clifford Pairs:* $\mathcal{V}(J)$ for a central-simple Clifford Jordan algebra J (a tight scalar form of some simple $J(\bar{q}, 1)$ for a nondegenerate quadratic form \bar{q} of dimension $\neq 2$ over an algebraically closed extension field $\bar{\Phi}$ of Φ).

All these classical pairs are *locally finite-dimensional*: every finitely generated subpair is finite-dimensional. For our ultimate use in (10.12) of an ultraproduct, we need to know more precisely that there is a *bound* (depending on the number of generators) for these dimensions. We say a system J has **locally d -bounded dimension** over Φ for a fixed function $d : N \rightarrow N$ if every subsystem $B \subset J$ generated over Φ by n elements is **spanned** over Φ by $d(n)$ elements (when Φ is a field this means $\dim_{\Phi}(B) \leq d(n)$).

Theorem 3.5. *All the central-simple classical pairs over Φ on the Clifford List 3.4 have locally d -bounded dimension for*

$$d(n) = (2n) 2^n (2n+1).$$

PROOF. We need to show that there is a function d such that any subpair B generated by n elements $(x_1, y_1), \dots, (x_n, y_n)$ (so B is the smallest subpair with B^+ containing x_1, \dots, x_n and B^- containing y_1, \dots, y_n) is contained in some subspace C spanned by $d = d(n)$ elements $(x_1, y_1), \dots, (x_d, y_d)$ (so $C^+ = \Phi x_1 + \dots + \Phi x_d$ and $C^- = \Phi y_1 + \dots + \Phi y_d$).

In rectangular pairs (I) $M_{p,q}(\Delta)$, $p \leq q$ the formulas $Q_{x^+}(y^-) = f(x^+, y^-)x^+$, $Q_{y^-}(x^+) = y^- f(x^+, y^-)$ show that $B \subset C = (R x_1 + \dots +$

$Rx_n, y_1R + \dots + y_nR$ for $R = M_{p,p}(\Delta)$, so $\dim_{\Phi}(C) = n \dim_{\Phi}(R)$; in (Ia) $R = M_{1,1}(\Phi)$ has dimension 1, in (Ib) $R = M_{1,1}(\mathcal{Q})$ has dimension 4, and in (Ic) $R = M_{2,2}(\Phi)$ has dimension 4, so in all cases $\dim(C) \leq 4n$.

In alternating pairs (II) $A_5(\Phi)$ we trivially have $B \subset C = V$ with $\dim(C) = 10$.

In Clifford pairs (III) $\mathcal{V}(J) = (J, J)$ for a simple $J = J(q, 1)$ in characteristic not 2, the formula $U_x y = q(x, \bar{y})x - q(x)\bar{y} = (t(x)t(y) - q(x, y))x - q(x)t(y)1 + q(x)y$ for $q(J) \subset \Phi$ shows $B \subset C = (K, K)$ for $K = \Phi 1 + \Phi x_1 + \dots + \Phi x_n + \Phi y_1 + \dots + \Phi y_n$ of dimension $2n + 1$, so $\dim(C) \leq 2n + 1$.

Thus in characteristic $\neq 2$ the pairs have locally d -bounded dimensions for $d(n) = \max(4n, 10)$. The characteristic 2 case is messier, and the bound $d(n) = (2n)2^{n(2n+1)}$ becomes enormous. Note that for all $n \geq 1$ we have $(2n)2^{n(2n+1)} \geq (2n)2^3 = 16n = 6n + 10n > 4n + 10 > \max(4n, 10)$, so $d(n)$ continues to serve as bound for all non-Clifford pairs in the List.

In characteristic 2, by [26, 6.5 p.163] the central-simple Clifford form J is contained in a full Clifford algebra $J(\tilde{q}, \tilde{\Phi})$ over a field $\tilde{\Phi}$ with $\Phi \subset \tilde{\Phi} \subset \bar{\Phi}$, $\tilde{\Phi}^2 \subset \Phi$. In particular, $\tilde{q}(J) \subset \tilde{\Phi}$ implies $\tilde{q}(J) + \tilde{q}(J, J) + \tilde{q}(J, \bar{J}) + \tilde{q}(\bar{J}) \subset \tilde{\Phi}$ (even though we don't know $\bar{J} \subset J$, we do have $\tilde{q}(\bar{J}) = \tilde{q}(J)$ and $\tilde{q}(J, \bar{J}) = \tilde{q}(J, 1)\tilde{q}(J, 1) - \tilde{q}(J, J)$), so

$$(1) \quad \tilde{q}(J)^2, \tilde{q}(\bar{J})^2, \tilde{q}(J, J)^2, \tilde{q}(\bar{J}, \bar{J})^2, \tilde{q}(J, \bar{J})^2 \subset \Phi.$$

Let

$$(2) \quad \Omega := \Phi[\tilde{q}(x_i), \tilde{q}(x_i, x_j), \tilde{q}(\bar{y}_i), \tilde{q}(\bar{y}_i, \bar{y}_j), \tilde{q}(x_i, \bar{y}_j)]$$

be the Φ -subalgebra of $\tilde{\Phi}$ generated by the $\tilde{n} = n + n(n-1)/2 + n + n(n-1)/2 + n^2 = 2n + n(n-1) + n^2 = 2n^2 + n = n(2n+1)$ different norms of x 's and \bar{y} 's. For uniformity we set

$$(3) \quad z_i := x_i, \quad z_{i+n} := \bar{y}_i \quad (1 \leq i \leq n)$$

so (2) simplifies to

$$(4) \quad \Omega = \Phi[\tilde{q}(z_i), \tilde{q}(z_i, z_j)] \quad (1 \leq i, j \leq 2n).$$

Introduce

$$(5) \quad \begin{aligned} K^+ &:= \sum_{i=1}^n \Omega x_i + \sum_{i=1}^n \Omega \bar{y}_i = \sum_{i=1}^{2n} \Omega z_i \\ K^- &:= \overline{K^+} = \sum_{i=1}^n \Omega \bar{x}_i + \sum_{i=1}^n \Omega y_i = \sum_{i=1}^{2n} \Omega \bar{z}_i \end{aligned}$$

so by definition (4) the norms of the generators fall in Ω and

$$(6) \quad \tilde{q}(K^+) = \tilde{q}(K^-) \subset \Omega.$$

This guarantees that $C = (K^+, K^-)$ is an Ω -subpair: $Q_{K^+}K^- \subset \sum \Omega Q_{z_i} \overline{z_k} + \sum \Omega Q_{z_i, z_j} \overline{z_k} = \sum \Omega [\tilde{q}(z_i, z_k)z_i - \tilde{q}(z_i)z_k] + \sum \Omega [\tilde{q}(z_i, z_k)z_j + \tilde{q}(z_j, z_k)z_i - \tilde{q}(z_i, z_j)z_k] \subset \sum \Omega z_i$ (by (6)) $= K^+$, applying bar gives $Q_{K^-}K^+ = \overline{Q_{K^+}K^-} \subset \overline{K^+} = K^-$. Since C is a subpair containing $(x_i, y_i) = (z_i, \overline{z_{i+n}})$ it contains the subpair they generate:

$$(7) \quad \begin{aligned} B &= \Phi[(z_1, \overline{z_{1+n}}), (z_2, \overline{z_{2+n}}), \dots, (z_n, \overline{z_{2n}})] \\ &\subset \left(\sum_{i=1}^{2n} \Omega z_i, \sum_{i=1}^{2n} \Omega \overline{z_i} \right) = C. \end{aligned}$$

Thus $\dim_{\Phi}(B) \leq \dim_{\Phi}(C) \leq \sum_{i=1}^{2n} \dim_{\Phi}(\Omega z_i) \leq (2n) \dim_{\Phi}(\Omega)$. Now Ω is generated by \tilde{n} elements $w_m = \tilde{q}(z_i)$ or $\tilde{q}(z_i, z_j)$ with $w_m^2 \in \Phi$ by (1), hence (as in Clifford or exterior algebras) Ω is spanned over Φ by $2^{\tilde{n}}$ monomials $w_{m_1} w_{m_2} \dots w_{m_r}$ for $m_1 < m_2 < \dots < m_r$, $0 \leq r \leq \tilde{n}$. Thus $\dim_{\Phi}(B) \leq (2n) 2^{\tilde{n}} = (2n) 2^{n(2n+1)} \leq d(n)$ as claimed in the theorem.

Thus there is an upper bound $d(n)$ for the dimension of n -generated subpairs, which is usually $4n$ but explodes for Clifford forms in characteristic 2. ■

4. Local Algebras

The Primitive Form Theorem in §5 asserts that a simple scalar form V of a classical pair \tilde{V} of Clifford type must be primitive, a crucial fact for classification purposes. The proof hinges on the “local-Cliffordness” of those classical pairs (all their local algebras are Clifford algebras). We now proceed to show that this is the case if \tilde{V} is one of the pairs of List 3.4. We begin with the rectangular case; for those, it amounts to carrying out the argument of example 1.9 of [7] for pairs. Recall that by (1.12.2) $Ker_J(b) = Ker(P_b)$ for nondegenerate (e.g. simple) systems.

Proposition 4.1. *If $V = \mathcal{V}(M, R, f)$ for a simple Artinian $R \simeq M_n(\Delta)$ and nondegenerate bilinear pairing f , then for each $b^- \in V^-$ there is an idempotent $e_b \in R$ with $f(M^+, b^-) = Re_b$, and*

$$A_b \simeq (e_b R e_b)^+ \simeq M_r(\Delta) \quad \text{where } r = \text{rank}(e_b).$$

PROOF. $f(M^+, b^-)$ is a left ideal in R , hence has the desired form Re_b . There are lots of choices for e_b , but notice that they all preserve b^- : $b^- = b^-e$ since $b^-(1-e) = 0$ by nondegeneracy of f , $f(M^+, b^-(1-e)) = f(M^+, b^-)(1-e) = Re(1-e) = 0$. We claim

$$(1) \quad V^{+(b^-)} \xrightarrow{\varphi} (e_b Re_b)^+ \text{ via } \varphi(x^+) = e_b f(x^+, b^-)$$

is an epimorphism with kernel precisely $\text{Ker}_J(b^-) = \text{Ker}(Q_{b^-})$, and so induces an isomorphism of A_b with $(e_b Re_b)^+$.

$$\varphi([f(x, b)x]) = ef([f(x, b)x], b) = ef(x, b)f(x, b) = ef(x, be)f(x, b)$$

First, φ is a Jordan homomorphism (using $be = b$): $\varphi(P_{x^+}^{(b)}y^+) = \varphi(Q_x Q_b y) = \varphi(f(x, Q_b y)x) = \varphi(f(x, bf(y, b))x) = \varphi(f(x, b)f(y, b)x) = ef(f(x, b)f(y, b)x, b) = e f(x, b) f(y, b) f(x, b) = e f(x, be) f(y, be)f(x, b) = ef(x, b)ef(y, b)ef(x, b) = \varphi(x) \varphi(y) \varphi(x)$, and also $\varphi(x^{+(2,b)}) = \varphi(Q_x b) = \varphi([f(x, b)x]) = ef([f(x, b)x], b) = ef(x, b)f(x, b) = ef(x, be)f(x, b) = ef(x, b)ef(x, b) = \varphi(x)^2$.

Second, φ is surjective: $e \in Re = f(M^+, b^-) \Rightarrow e = f(c^+, b^-)$ for some $c^+ \in M^+ \Rightarrow \varphi(rc^+) = ef(rc^+, b^-) = erf(c^+, b^-) = ere$.

Finally, the kernel of φ is the kernel of Q_b : if $ef(x, b) = 0$ then $Q_b x = bf(x, b) = (be)f(x, b) = 0$, and conversely if $Q_b x = 0$ then $0 = f(c, Q_b x) = f(c, bf(x, b)) = f(c, b) f(x, b) = ef(x, b)$. ■

Corollary 4.2. *If a central-simple pair $V = \mathcal{V}(M, R, f)$ of finite capacity satisfies a Clifford HPI as in (3.4), then its local algebras have dimension 1 or 4 as follows:*

- (Ia) $V = \mathcal{M}_{1,q}(\Phi)$: if $b^- \neq 0$, $A_{b^-} \simeq \Phi^+$,
- (Ib) $V = \mathcal{M}_{1,q}(\mathcal{Q})$: if $b^- \neq 0$, $A_{b^-} \simeq \mathcal{Q}^+$, \mathcal{Q} a quaternion division algebra,
- (Ic) $V = \mathcal{M}_{2,q}(\Phi)$: if $\text{rank}(b^-) = 1$, $A_{b^-} \simeq \Phi^+$,
if $\text{rank}(b^-) = 2$, $A_{b^-} \simeq M_2(\Phi)^+$.

PROOF. In (3.4) (Ia) and (Ib) the only nonzero e_b is $e = 1$, so $A_{b^-} \simeq (e_b Re_b)^+ = R^+$. In (Ic), if b^- has rank 1 then so does e_b , and $A_{b^-} \simeq (e_b Re_b)^+ = \Phi e_b^+ \simeq \Phi^+$; if b^- has rank 2 then $e_b = 1$, $A_{b^-} \simeq R^+$. ■

We now turn to the alternating pair $V = \mathcal{V}(A_5(\Phi))$ over a field Φ . We think of V as the alternating linear maps between a 5-dimensional Φ -space W and its dual W^* as in (1.8.3). We have

Proposition 4.3. *If $V = \mathcal{V}(A_5(\Phi)) = (Alt(W, W^*), Alt(W^*, W))$ as in (1.8.3), then a local algebra of V is isomorphic to either Φ^+ of dimension 1, or $J(q, 1)$ of dimension 6.*

PROOF. An element $b^- \in Hom(W^*, W)$ has even (matrix) rank 0, 2 or 4 ($\text{rank}(b) = 0 \Rightarrow b = 0$). If $\text{rank}(b) = 2$, then there exists some basis \mathcal{B} of W such that the matrix $B = Mat_{\mathcal{B}, \mathcal{B}^*}(b)$ of b with respect to \mathcal{B} and its dual \mathcal{B}^* is

$$B = \text{diag} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right\}$$

and $Ker_J(b) = Ker(P_b) =$ all alternating maps $x^+ \in Hom(W, W^*)$ whose 5×5 matrix $X = Mat_{\mathcal{B}, \mathcal{B}^*}(x^+)$ has $X_{11} = X_{12} = X_{21} = X_{22} = 0$, so that $A_b = \Phi \tilde{c}$, where $c \in Hom(W, W^*)$ has the same matrix $Mat_{\mathcal{B}, \mathcal{B}^*}(c) = B$; since $Mat_{\mathcal{B}, \mathcal{B}^*}(-cbc) = -BBB = B = Mat_{\mathcal{B}, \mathcal{B}^*}(c)$ we have $\tilde{c}^2 = \widetilde{P_c b} = -\widetilde{cb}c = \tilde{c}$ which shows that $A_b \simeq \Phi^+$.

If $\text{rank}(b) = 4$, then some basis of W yields the matrix representation

$$Mat_{\mathcal{B}, \mathcal{B}^*}(b) = B := \text{diag} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0 \right\}$$

in which case $Ker(P_b) =$ all alternating maps x^+ whose 5×5 matrix $X = Mat_{\mathcal{B}, \mathcal{B}^*}(x^+)$ has $X_{ij} = X_{55} = 0$, $1 \leq i, j \leq 4$, and

$$\begin{aligned} A_b &\simeq \left\{ \begin{pmatrix} \alpha s & a \\ -a^t & \beta s \end{pmatrix} \mid s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, a \in M_2(\Phi), \alpha, \beta \in \Phi \right\} \\ &\simeq A_4(\Phi) \simeq H_2(\mathcal{Q}(\Phi), \Phi, s) \\ &\simeq \left\{ \text{all} \begin{pmatrix} \alpha & a \\ \bar{a} & \beta \end{pmatrix} \right\} = J(q, 1) \end{aligned}$$

for $q \left(\begin{pmatrix} \alpha & a \\ \bar{a} & \beta \end{pmatrix} \right) = \det \begin{pmatrix} \alpha & a \\ \bar{a} & \beta \end{pmatrix} = \alpha\beta - a\bar{a} \in \Phi$, $\bar{a} = sa^t s^{-1}$ the quaternion involution, and q is nondegenerate of dimension $1 + 1 + 4 = 6$. \blacksquare

Finally, in the case of Clifford Jordan pairs, the local algebras are again Clifford forms by [23, 6.2(ii)]. We have

Proposition 4.4. *If $V = \mathcal{V}(J)$ for a central-simple Clifford Jordan algebra J (a tight scalar form of some simple $J(\bar{q}, 1)$ for a nondegenerate \bar{q} of dimension $\neq 2$ over an algebraically closed extension field $\bar{\Phi}$ of Φ), then all the local algebras of V are Clifford forms: they have tight scalar extensions $\bar{A} = \bar{\Phi}A \supset A$ which are again simple $J(\bar{q}, 1)$'s of dimension $\neq 2$ over $\bar{\Phi}$, more precisely are either*

- (i) *tiny $\bar{\Phi}^+$ of dimension 1,*
- (ii) *small $H_2(\bar{\Phi})$ [or $H(\mathcal{Q}(\bar{\Phi}), -)$, in characteristic 2 only] of dimension 3,*
- (iii) *large $J(\bar{q}, \bar{1})$ of dimension ≥ 3 (actually ≥ 4 in characteristic 2).*

PROOF. If J itself were a full Clifford Jordan algebra $J(q, 1)$, as in characteristic $\neq 2$, we could give an easy direct proof that the local algebras are *full Clifford algebras*, but in characteristic 2 the matter is more complex, and we have to settle for (perhaps non-full) Clifford *forms* instead of full Clifford algebras $J(q, 1)$. For this we fall back on [23, Thm. 6.2]. In all cases the local A_b 's are at least Clifford forms, hence (by [26, 6.1 p.160], [23, 3.5]) are tight scalar forms of simple $J(\bar{q}, \bar{1})$'s of dimension $\neq 2$. ■

Theorem 4.5 (Local Classical Algebras). *If V is a classical central-simple Jordan pair from the Clifford List 3.4, then all its local algebras A_b for $b \neq 0$ are Clifford forms: they are full Clifford $J(q, 1)$ of dimensions 1, 4 or 6, unless V is itself a simple Clifford form $\mathcal{V}(J)$. In all cases they have tight scalar extensions $\bar{A} \simeq J(\bar{q}, \bar{1})$ of dimension $\neq 2$ over the algebraically closed field $\bar{\Phi}$.*

PROOF. The Clifford List 3.4 is closed under forming opposites, so if all V_{b^-} are Clifford so are all V_{b^+} , and it suffices to check the A_b for $b = b^-$. Let $0 \neq b \in V^-$ for a V from the list. Using Cor. 4.2, if V is $\mathcal{M}_{1,q}(\Phi)$ of type (Ia) then all A_b are Φ^+ of dimension 1; if V is $\mathcal{M}_{1,q}(\mathcal{Q})$ of type (Ib) (\mathcal{Q} a quaternion division algebra over Φ) they are all $\mathcal{Q}^+ \simeq J(q, 1)$ of dimension 4; if V is $\mathcal{M}_{2,q}(\Phi)$ of type (Ic) then all A_b are Φ^+ of dimension 1 or $\mathcal{M}_2(\Phi)^+ \simeq J(q, 1)$ of dimension 4. Using Prop. 4.3, if V is $A_5(\Phi)$ the A_b are $J(q, 1)$'s of dimension 1 or 6. Only the case $V = \mathcal{V}(J)$ for a simple Clifford form J is messy and we appeal to 4.4. ■

5. Primitive Form Theorem

In this section we will examine simple pairs V that have a tight scalar extension $\tilde{V} = V\tilde{\Phi}$ of classical type. In characteristic $\neq 2$ one usually takes ordinary tensor products $\tilde{V} = V \otimes_{\Phi} \tilde{\Phi}$; when the centroid Φ contains $1/2$ then V is outer-simple, and the usual argument for associative algebras (using the Jacobson Density Theorem for V as an irreducible $\mathcal{M}(V)$ -module, \mathcal{M} the multiplication algebra of V) shows \tilde{V} is again (outer-) simple over $\tilde{\Phi}$. The directness of the imbedding of V in $V \otimes_{\Phi} \tilde{\Phi}$ makes it fairly easy to show that if \tilde{V} is of a certain type over $\tilde{\Phi}$, then V is of the same type over Φ . This argument breaks down in characteristic 2: V need not be outer-simple, and even worse, tensor products $V \otimes_{\Phi} \tilde{\Phi}$ over the centroid need not remain simple. So we replace free scalar extensions with tight ones (1.15) (any scalar extension, e.g. the free $V \otimes_{\Phi} \tilde{\Phi}$, can be tightened by dividing out by an ideal \tilde{I} maximal with respect to $\tilde{I} \cap V = 0$).

We begin by deriving a series of lemmas concerned with how scalars $\tilde{\omega} \in \tilde{\Phi}$ in a pair \tilde{V} over $\tilde{\Phi}$ interact with a central simple Φ -subpair V .

Lemma 5.1. *If x (resp. $x^\varepsilon \in V^\varepsilon$) is any (resp. homogeneous) element of a Jordan triple system or algebra J (resp. Jordan pair V) then $I_x = \mathcal{M}(J)P_xJ$ (resp. $I_{x^\varepsilon} = \mathcal{M}(V)Q_{x^\varepsilon}V^{-\varepsilon}$) is an ideal in J (resp. V) while $I_x^\circ = \mathcal{M}(J)x$ (resp. $I_{x^\varepsilon}^\circ = \mathcal{M}(V)x^\varepsilon$) is an outer ideal; if J (resp. V) is simple (hence nondegenerate) then $I_x = J$ for all $x \neq 0$ (resp. $I_{x^\varepsilon} = V$ for all $x^\varepsilon \neq 0$), and if J (resp. V) is outer-simple (= simple when $1/2 \in \Phi$) then $I_x^\circ = J$ for all $x \neq 0$ (resp. $I_{x^\varepsilon}^\circ = V$ for all $x^\varepsilon \neq 0$).*

PROOF. Clearly I_x° (resp. $I_{x^\varepsilon}^\circ$) is a (resp. polarized) outer ideal since this just means invariant under \mathcal{M} ; if the system is outer-simple and $x \neq 0$ this outer ideal must be the whole system. The harder part concerns the quadratic expression for I_x (note that we aren't trying to find an expression for the ideal generated by x ; that would be $\mathcal{M}P_xJ + \mathcal{M}x$ [in the algebra case we would have to add $\mathcal{M}x^2$ as well]). Since nonzero elements in a simple system can't be absolute zero-divisors, the (resp. polarized) space I_x is nonzero for $x \neq 0$; once it is an ideal it will be all of J (resp. V) in the simple case.

I_x is by construction an outer ideal spanned by elements MP_xy for

structural M 's (products of $Id, P_z, B_{z,w}$ which in the pair case can be taken as $Q_{z^\varepsilon}, B_{z^\varepsilon, w^{-\varepsilon}}$, and in the algebra case $B_{z,1}$ as well). Once $P_c J \subset I_x$ for a spanning set of c 's for the outer ideal I_x , I_x is an inner ideal as well, so we only need to prove innerness $P_c a \in I_x$ for $a \in J$ (resp. $a \in \hat{J}$ in the algebra case), $c = MP_x y \in I_x$ (resp. $c \in I_{x^\varepsilon}^\eta$, $a \in V^{-\eta}$ in the pair case). But by structurality of M we have $P_c a = MP_x P_y P_x M^* a \in MP_x J$ (even if $a \in \hat{J}$ in the algebra case). ■

This will allow us to show that if $\tilde{\omega}x$ lies in a simple system J (for $\tilde{\omega}$ a scalar acting on some larger system $\tilde{J} \supset J$ over $\tilde{\Phi} \supset \Phi$) then $\tilde{\omega}^2$ lies in the centroid Φ : P_x absorbs $\tilde{\omega}^2$ (but not $\tilde{\omega}$) to produce a multiplication in J , $\tilde{\omega}^2 P_x = P_{\tilde{\omega}x} \in \mathcal{M}(J)$. It is clear that if $J \subset \tilde{J}$ and $\tilde{T} \in \Gamma(\tilde{J})$ leaves J invariant, then the restriction $T = \tilde{T}|_J$ belongs to $\Gamma(J)$.

Lemma 5.2. *Let J be a simple Jordan triple system (resp. algebra, pair) over its centroid Φ , and assume $J \subset \tilde{J}$ for a system over a field $\tilde{\Phi} \supset \Phi$.*

- (i) *If some $\tilde{\omega} \in \tilde{\Phi}$ multiplies a single (resp. homogeneous) element of J back into J , then $\tilde{\omega}^2 \in \Phi$ (or: $\tilde{\omega} \in \Phi$ if J is outer-simple):*
 $0 \neq \tilde{\omega}x \in J$ for $x \in J$ (resp. $x^\varepsilon \in V^\varepsilon$) $\implies \tilde{\omega}^2 \in \Phi$ (or: $\tilde{\omega} \in \Phi$).
- (ii) *If some $\tilde{\omega} \in \tilde{\Phi}$ multiplies a local element of A_y back into A_y for $y \in J$ (resp. $y \in V^-$) then $\tilde{\omega}^2 \in \Phi$ (or: $\tilde{\omega} \in \Phi$ if J is outer-simple):*
 $0 \neq \tilde{\omega}a = b \in A_y$ for $a \in A_y$ $\implies \tilde{\omega}^2 \in \Phi$ (or: $\tilde{\omega} \in \Phi$).
- (iii) *Scalar-regularity breeds regularity: if some $x \in J$ (resp. $x^\varepsilon \in V^\varepsilon$) is scalar-regular via $z \in J$ (resp. $z^- \in V^-$), or some a in a local algebra A_y for $y \in J$ (resp. $y^- \in V^-$) is scalar-regular via $b \in A_y$, then that scalar multiple is regular in J or A_y ,*
 $P_x z = \tilde{\omega}x \neq 0$ (resp. $z^- \in V^-$) $\implies \tilde{\omega}x$ regular in J , or
 $U_a b = \tilde{\omega}a \neq 0 \implies \tilde{\omega}a$ regular in A_y .
- (iv) *If $x \in J$ has $0 \neq P_x^n \in \Gamma(\tilde{J})$ for some $n \geq 1$, then x is invertible (hence regular) in J .*
- (v) *If J is radical then neither J nor any of its local algebras A_y can have scalar-regular elements.*

PROOF. For (i), it suffices to prove $\tilde{\omega}^2 J \subset J$ (or: $\tilde{\omega}J \subset J$). But by (5.1 (1)) $J = \mathcal{M}P_x J$ so $\tilde{\omega}^2 J = \mathcal{M}P_{\tilde{\omega}x} J \subset J$ (or: $J = \mathcal{M}x$, so $\tilde{\omega}J = \mathcal{M}(\tilde{\omega}x) \subset J$).

For (ii), let us first recall $Ker_J(y) = Ker(P_y)$ by (1.12.2). Now, if $a = \bar{x}, b = \bar{z}$ for $x, z \in J$ (resp. $x, z \in V^+$) and $\tilde{\omega}a = b$ then $\tilde{\omega}x = z + \tilde{k}$ ($z \in$

J , $\tilde{k} \in \text{Ker}(\tilde{P}_y)$ in \tilde{J}). Therefore $\tilde{\omega}\tilde{P}_y x = P_y z + 0$, so $\tilde{\omega}(P_y x) = P_y z \neq 0$ is in J (resp. V^-) [if $P_y z = 0$ then $z \in \text{Ker}(P_y)$, $b = \bar{z} = \bar{0}$ in A_y , whereas by hypothesis $b \neq 0$]. Thus by (i), $\tilde{\omega}^2 \in \Phi$ (or $\tilde{\omega} \in \Phi$ in the outer-simple case).

For (iii), from $P_x z = \tilde{\omega}x \neq 0$ we get $0 \neq \tilde{\omega}^2 = \alpha \in \Phi$ by (i), where α is invertible in Φ , since Φ is a field by simplicity of J . Thus $P_{\tilde{\omega}x} J = \tilde{\omega}^2 P_x J = P_x J$ ($\alpha J = J$) $\ni P_x z = \tilde{\omega}x$ and $\tilde{\omega}x$ is regular. Similarly $0 \neq \tilde{\omega}a = U_a b \in A = A_y \Rightarrow \tilde{\omega}^2 = \alpha \in \Phi$ (by (ii)) $\Rightarrow U_{\tilde{\omega}a} A = \tilde{\omega}^2 U_a A = U_a(\alpha A) = U_a(A) \ni U_a b = \tilde{\omega}a$ so $\tilde{\omega}a$ is regular in A_y .

For (iv) $0 \neq P_x^n \in \Gamma(\tilde{J}) \cap \text{End}(J) \subset \Gamma(J) = \Phi$ a field implies P_x^n (so also P_x and x) invertible on J .

Finally, (v) follows from (iii): if J is radical, so is any homotope $J^{(b)}$ and any local algebra A_b , and radical systems can't contain nonzero regular elements. \blacksquare

Note in (iv) that if J is a polarized triple and $x = x^+ \oplus x^-$, then odd powers of $P_x = \begin{pmatrix} 0 & P_{x^+} \\ P_{x^-} & 0 \end{pmatrix}$ are never central, but powers of $P_x P_x = \begin{pmatrix} P_{x^+} P_{x^-} & 0 \\ 0 & P_{x^-} P_{x^+} \end{pmatrix}$ will be central if $P_{x^\varepsilon} P_{x^{-\varepsilon}} = \tilde{\gamma} \text{Id}_{V^\varepsilon}$ for some $\tilde{\gamma} \in \tilde{\Gamma}$.

It will be important in (5.5) that we allow P_x^n to be in $\Gamma(\tilde{J})$ (which may not be a field), not just in the field $\tilde{\Phi}$.

Our ultimate goal is to show that central simple forms of classical pairs of Clifford type are actually primitive. The next three lemmas aim at showing that such forms (along with their local algebras) contain regular elements and therefore cannot be radical (in the sense of Jacobson). By (5.2 (iii)), it will be enough to establish the existence of scalar-regular elements.

Lemma 5.3. *If A is a nondegenerate Jordan Φ -subalgebra of a Clifford Jordan algebra $\tilde{A} = J(\tilde{q}, \tilde{1})$ of a quadratic form \tilde{q} over a field $\tilde{\Phi} \supset \Phi$, an element $a \neq 0$ is scalar-regular in A (i.e. some $0 \neq \tilde{\omega}a \in U_a A$) if and only if either*

- (i) $\tilde{q}(a) = 0$, or
- (ii) $\tilde{q}(a) \neq 0$ and $\bar{a} = \tilde{\mu}^{-1}b$ for some $\tilde{\mu} \in \tilde{\Phi}$, $b \in A$ (in which case $0 \neq U_a b^2 \in A \cap \tilde{\Phi}1$).

In particular,

- (iii) $a \neq 0$ is scalar-regular if it is symmetric or skew ($\bar{a} = \pm a$, e.g. if $\tilde{t}(a) = 0$).

(iv) A will have scalar-regular elements unless \tilde{q}, \tilde{t} are anisotropic on it, it is asymmetric, and all its inner derivations vanish, $\mathcal{D}_{A,A} = 0$.

PROOF. (i) If $\tilde{q}(a) = 0$ then $U_a A = \tilde{q}(a, \bar{A})a \subset \tilde{\Phi}a$; by nondegeneracy $U_a A \neq 0$ (unless $a = 0$, which is trivially regular) so $\tilde{q}(a, \bar{b}) = \tilde{\omega} \neq 0$ for some $b \in A$ and some $\tilde{\omega} \in \tilde{\Phi}$, and $U_a b = \tilde{\omega}a \neq 0$.

(ii) If $\tilde{q}(a) \neq 0$ then $U_a b = \tilde{q}(a, \bar{b})a - \tilde{q}(a)\bar{b}$ is of the form $\tilde{\omega}a$ iff $(\tilde{q}(a, \bar{b}) - \tilde{\omega})a = \tilde{q}(a)\bar{b}$ iff $\bar{b} = \tilde{\mu}a \in \bar{A}$ for some $\tilde{\mu}$ (since $\tilde{q}(a)$ is invertible in $\tilde{\Phi}$) iff $b = \tilde{\mu}\bar{a} \in A$. Here $\tilde{\omega}a \neq 0$ iff $\tilde{\mu} \neq 0$: $\tilde{\mu} = 0 \implies b = 0 \implies \tilde{\omega}a = U_a b = 0$ and $\tilde{\omega}a = 0 \implies 0 = U_a b = \tilde{\mu}U_a \bar{a} = \tilde{\mu}\tilde{q}(a)a \implies \tilde{\mu} = 0$. Thus $U_a b = \tilde{\omega}a \neq 0$ iff $\bar{a} = \tilde{\mu}^{-1}b$ for some $\tilde{\mu} \neq 0$ in $\tilde{\Phi}$. When $b = \tilde{\mu}\bar{a}$ we have $U_a b^2 = \tilde{\mu}^2 U_a \bar{a}^2 = \tilde{\mu}^2 \tilde{q}(a)^2 1 \in \tilde{\Phi}1$ nonzero by $\tilde{\mu}, \tilde{q}(a) \neq 0$.

(iii) If a is symmetric or skew then a is scalar-regular automatically by (i) if $\tilde{q}(a) = 0$, and by (ii) if $\bar{a} = \pm a$ where $\tilde{q}(a) \neq 0$.

Finally, (iv) follows from (i), (iii) since inner derivations produce skew elements. ■

So far A was an arbitrary nondegenerate subalgebra. In the case we are interested in, A is a *scalar form* of a Jordan algebra of a nondegenerate quadratic form (in which case A is *automatically* nondegenerate).

Lemma 5.4. *Let A be a scalar form ($A\tilde{\Phi} = \tilde{A}$) of a Clifford Jordan algebra $\tilde{A} = J(\tilde{q}, \tilde{1})$ for a nondegenerate quadratic form \tilde{q} over a field $\tilde{\Phi} \supset \Phi$. If A contains no scalar-regular elements then \tilde{A} is 2-dimensional over $\tilde{\Phi}$ (if \tilde{A} is a tight extension of A , then $\tilde{A} = \tilde{\Omega}^+$ for a degree 2 field extension $\tilde{\Omega}$ of $\tilde{\Phi}$) with \tilde{q}, \tilde{t} anisotropic on A .*

PROOF. Assume A contains no scalar-regular elements. Since A is nondegenerate, we can apply (5.3 (iv)) to conclude \tilde{q}, \tilde{t} are anisotropic on A , and A is asymmetric without inner derivations. Then $\mathcal{D}_{A,A}A = 0 \implies \mathcal{D}_{\tilde{A},\tilde{A}}\tilde{A} = 0$ too since \tilde{A} is a scalar extension of A .

Now \tilde{t} anisotropic certainly implies $\tilde{t} \neq 0$; but for a nondegenerate traceful form this forces $\dim_{\tilde{\Phi}}(\tilde{A}) \leq 2$. Indeed, if $\dim_{\tilde{\Phi}}(\tilde{A}) \geq 3$ then for any $\tilde{x} \in \tilde{A}$, $\tilde{A} \not\subset \tilde{\Phi}1 + \tilde{\Phi}\tilde{x}$, yet by [23, (2.2.1)] $\mathcal{D}_{\tilde{x},\tilde{A}}\tilde{x} = 0 \implies [4\tilde{q}(\tilde{x}) - \tilde{t}(\tilde{x})^2]\tilde{A} \subset \tilde{\Phi}1 + \tilde{\Phi}\tilde{x}$, so for all \tilde{x} we would have

$$4\tilde{q}(\tilde{x}) - \tilde{t}(\tilde{x})^2 = 0.$$

In characteristic 2 any $x \neq 0$ in A has $4\tilde{q}(x) - \tilde{t}(x)^2 = \tilde{t}(x)^2 \neq 0$ by anisotropy

of the trace on A . In characteristic $\neq 2$ (assuming the dimension is at least 2) by nondegeneracy of \tilde{q} we can find $\tilde{x} \in \tilde{A}$ with $\tilde{t}(\tilde{x}) = 0 \neq \tilde{q}(\tilde{x})$, again leading to a contradiction $4\tilde{q}(\tilde{x}) - \tilde{t}(\tilde{x})^2 = 4\tilde{q}(\tilde{x}) \neq 0$. Thus the assumption that the dimension is 3 or more is untenable.

On the other hand, if $\dim_{\tilde{\Phi}}(\tilde{A}) = 1$ then $A \subset \tilde{A} = \tilde{\Phi}1$ is entirely symmetric, contrary to asymmetry. Thus \tilde{A} can only be 2-dimensional over $\tilde{\Phi}$.

It is “well-known” that \tilde{A} is either a degree 2 field extension or is split,

$$(5.4.1) \quad \begin{aligned} & \text{a nondegenerate } J(\tilde{q}, \tilde{1}) \text{ of dimension 2 over a field } \tilde{\Phi} \\ & \text{is either } (i) \tilde{\Omega}^+ \text{ for a field extension } \tilde{\Omega} \text{ of } \tilde{\Phi}, \text{ or} \\ & (ii) \text{ a split } \tilde{\Phi}\tilde{e}_1 \oplus \tilde{\Phi}\tilde{e}_2, \end{aligned}$$

and that case (i) never arises if $\tilde{\Phi}$ is algebraically closed. Indeed, for any $x \notin \tilde{\Phi}1$ we have $\tilde{A} = \tilde{\Phi}[x] \cong \tilde{\Omega}^+$, $\tilde{\Omega} = \tilde{\Phi}[x]/I$ for an associative ideal I (I is a Jordan ideal, and if $z \in I, f \in \tilde{\Phi}[x]$ then $zf \in I$ by nondegeneracy of \tilde{A} since $U_{zf}\tilde{\Phi}[x] = U_zU_f\tilde{\Phi}[x] \subseteq I$). Either \tilde{A} is a division algebra ($\tilde{\Omega}$ is a field), or there exist $0 \neq \tilde{z} \in \tilde{A}$ with $\tilde{q}(\tilde{z}) = 0$, in which case $\tilde{z}^2 \neq 0, \tilde{t}(\tilde{z}) \neq 0$ since $\tilde{\Omega}$ has no nilpotent elements by nondegeneracy, so $\tilde{e} := \tilde{t}(\tilde{z})^{-1}\tilde{z}$ has $\tilde{t}(\tilde{e}) = 1, \tilde{q}(\tilde{e}) = 0, \tilde{e}$ is an idempotent, and $\tilde{A} = \tilde{\Phi}\tilde{e} \boxplus \tilde{\Phi}(\tilde{1} - \tilde{e})$ is split.

But if the tight extension \tilde{A} is split, then A automatically has scalar-regular elements: there are $0 \neq a = \tilde{\omega}\tilde{e}_1 \in A \cap \tilde{\Phi}\tilde{e}_1$ with $U_a a = \tilde{\omega}^3\tilde{e}_1 = \tilde{\omega}^2 a$. If there are no scalar-regular elements, the tight extension \tilde{A} can only be a field extension of a non-algebraically-closed field $\tilde{\Phi}$. ■

We now give an example of $A \subset \tilde{A} \simeq \tilde{\Omega}^+$ for a 2-dimensional field extension $\tilde{\Omega}$ of $\tilde{\Phi}$ where \tilde{q}, \tilde{t} are anisotropic and A is prime but has no $\tilde{\Phi}$ -regular elements at all.

Example 5.4.2. *Let $\tilde{\Omega}$ be a quadratic extension field of $\tilde{\Phi}$ and $a \in \tilde{\Omega}$ with $t = \tilde{t}(a), q = \tilde{q}(a)$ algebraically independent over the field $\tilde{\Phi}$. [For example, take $\tilde{\Omega}, \tilde{\Phi}, \Phi$ to be the complexes, reals and rationals, $a = \tau + i\sigma$ with real τ, σ algebraically independent over the rationals, so $t = 2\tau, q = \tau^2 + \sigma^2$ are also algebraically independent over the rationals.] Then $A = \Phi[a]_0 = \sum_{i=1}^{\infty} \Phi a^i$ is an infinite-dimensional domain over Φ which is a scalar form of $\tilde{\Phi}A = \tilde{A} = \tilde{\Omega}$ a field, yet A contains no nonzero $\tilde{\Phi}$ -scalar-regular, $\tilde{\Phi}$ -invertible, or quasi-invertible elements at all.*

PROOF SKETCH. Note that a is not a scalar ρ in $\tilde{\Phi}1$, since then $t = 2\rho$,

$q = \rho^2$ would be algebraically dependent. An easy induction from $a = 1 \cdot a - 0 \cdot 1$, $a^2 = ta - q1$ shows $a^n = t_{n-1}a - q_n1$ where the $\{t_n\}_{n \geq 0}$ are congruent to $t^n \pmod q$, so are linearly independent over Φ , hence the $\{a^n\}_{n \geq 1}$ are linearly independent over Φ modulo $\tilde{\Phi}1$; by usual results on polynomials with zero constant terms, A has no scalar-invertible or quasi-invertible elements ($b, c \neq 0$ with $bc = \tilde{\alpha}1$ or $bc = b + c$), hence it has no scalar-regular elements ($0 \neq \tilde{\omega}a = U_a b = a^2 b \implies \tilde{\omega}1 = ab$ by cancelling $a \in \tilde{\Omega}$). ■

Even if A has $\tilde{\Phi}$ -regular elements, we do not have simplicity of A or the fact that the centroid is a field, which is used to get regular elements in A from $\tilde{\Phi}$ -regular elements. We now exhibit examples of algebras A which have no regular elements, but may still have scalar-regular elements. It will be easier to see what's going on in a (non-tight) split extension \tilde{A} .

Example 5.4.3. (i) *An element with injective U -operator is regular iff it is invertible, in which case A must be unital. A Jordan domain has no nonzero regular elements iff it is not unital.*

(ii) *If $A \subseteq \tilde{A}$ with \tilde{A} unital over $\tilde{\Phi}$, and $U_a b = \tilde{\lambda}a$ for $a, b \in A$, $\tilde{\lambda} \in \tilde{\Phi}$ with $U_a, \tilde{\lambda}$ injective on \tilde{A} , then $U_a b^2 = \tilde{\lambda}^2 \tilde{1}$, $a \circ b = 2\tilde{\lambda} \tilde{1} \in A$ and $\tilde{\lambda}^2 I, 2\tilde{\lambda} I \in \text{Centroid}(A)$. An A acting injectively on \tilde{A} over a field $\tilde{\Phi}$ has no nonzero $\tilde{\Phi}$ -regular elements iff $A \cap \tilde{\Phi} \tilde{1} = 0$.*

(iii) *Let $\tilde{A} := \tilde{\Phi}e_1 \oplus \tilde{\Phi}e_2$ be a split $J(\tilde{q}, \tilde{1})$ of dimension 2 over a field $\tilde{\Phi}$, and let $\Phi_0 \subseteq \tilde{\Phi}$ be a Jordan Φ -subalgebra with a Φ -linear Jordan monomorphism $\Phi_0 \xrightarrow{\sigma} \tilde{\Phi}$. Then $A := \{a(\alpha) := \alpha e_1 \oplus \sigma(\alpha)e_2 \mid \alpha \in \Phi_0\} \cong \Phi_0^+$ is a Jordan domain. The norm \tilde{q} is always anisotropic on A , and the trace \tilde{t} is anisotropic on A iff σ has no nonzero skew elements $\sigma(\alpha) = -\alpha$ (e.g. if it has no fixed points); A is a $\tilde{\Phi}$ -form of \tilde{A} (necessarily non-tight) iff σ is nontrivial; A has no regular elements iff $1 \notin \Phi_0$; A has no $\tilde{\Phi}$ -regular elements iff σ has no nonzero fixed points.*

PROOF SKETCH. (i) By the Fundamental Formula and cancellation $U_a b = a \implies U_a(U_b U_a - I) = 0 \implies U_b U_a = I$; applying to b then gives $U_b a = b$, so by symmetry $U_a U_b = U_b U_a = I \implies U_a$ invertible, which is well-known to imply $1 = U_a b^2 \in J$. (ii) $U_a b = \tilde{\lambda}a \implies \tilde{U}_a(\tilde{U}_b \tilde{U}_a - \tilde{\lambda}^2 \tilde{1}) = 0$ on $\tilde{A} \implies \tilde{U}_b \tilde{U}_a = \tilde{\lambda}^2 \tilde{1}$; applying to b and cancelling $\tilde{\lambda}$ then gives $U_b a = \tilde{\lambda}b$, so by symmetry $\tilde{U}_a \tilde{U}_b = \tilde{U}_b \tilde{U}_a = \tilde{\lambda}^2 \tilde{1} \in \text{Centroid}(A) \implies \tilde{\lambda}^2 \tilde{1} = \tilde{U}_a \tilde{U}_b(\tilde{1}) = U_a(b^2) \in A$. Similarly $\tilde{U}_a \tilde{U}_b(\tilde{V}_{a,b} - 2\tilde{\lambda} \tilde{1}) = 0 \implies \tilde{V}_{a,b} = 2\tilde{\lambda} \tilde{1} \in \text{Centroid}(A) \implies 2\tilde{\lambda} \tilde{1} =$

$$\tilde{V}_{a,b}(\tilde{1}) = a \circ b \in A.$$

(iii) Clearly $a : \Phi_0^+ \rightarrow A$ is a Jordan isomorphism with inverse $A \xrightarrow{proj} \Phi_0 e_1 \cong \Phi_0^+$, in particular A is a domain acting injectively on \tilde{A} ; $\tilde{q}(a(\alpha)) = \alpha\sigma(\alpha) = 0$ iff $\alpha = 0$ since σ is injective; $\tilde{t}(a(\alpha)) = \alpha + \sigma(\alpha) = 0$ iff $\sigma(\alpha) = -\alpha$ (which leads to a fixed point $\sigma(\alpha^2) = +\alpha^2$). If $\sigma = 1$ is trivial then $A = \Phi_0 1$ is a form of $\tilde{\Phi} 1$, not \tilde{A} , but if σ is nontrivial then for some α we have $0 \neq (\sigma(\alpha) - \alpha)\sigma(\alpha)e_2 = a(\alpha^2) - \alpha a(\alpha) \in \tilde{\Phi} A \Rightarrow e_2 \in \tilde{\Phi} A \Rightarrow \tilde{\Phi} A = \tilde{A}$ and A is a form (it is never tight since $A \cap \tilde{\Phi} e_2 = 0$). By (i) A has no regular elements iff $1 \notin A$, equivalently $1 \notin \Phi_0$ (since $\sigma(1)$ would have to be 1). By (ii) A has no scalar-regular elements iff $A \cap \tilde{\Phi} \tilde{1} = 0$, equivalently iff σ has no nonzero fixed points ($a(\alpha) = \tilde{\lambda} \tilde{1} \Leftrightarrow \alpha = \tilde{\lambda} = \sigma(\alpha) \Leftrightarrow \sigma(\alpha) = \alpha$). ■

For example, let $\tilde{\Phi} = \Omega(t)$ or $\Omega(t, s)$ be the field of rational functions in one or two variables over a field Ω , and $\Phi_0 = \Omega[t]_0$ the polynomials in t with zero constant term. Then Φ_0 certainly doesn't contain 1, so A has no regular elements. It will have no scalar-regular elements if σ has no fixed points, for example if $\sigma(f(t)) = f(t^n)$ for $n > 1$ or [in the two-variable case] $\sigma(f(t)) = f(s)$. But A will have scalar-regular-but-not-regular elements in case σ does have fixed points, for example if $\sigma(f(t)) = f(\xi t)$ for $1 \neq \xi \in \Omega$ a primitive n^{th} root of unity ($n > 1$ not divisible by the characteristic of Ω) with fixed subalgebra $\Omega[t^n]_0 \subset \Phi_0$. [In the two-variable case $\tilde{\Phi} = \Omega(t, s)$ with $\Phi_0 = \Omega[t, s]_0$ the map $\sigma(f(t, s)) = f(s, t)$ is another example with fixed-points the symmetric polynomials with zero constant term.]

Replacing polynomials by power series $\tilde{\Phi}_0 = \Omega[[t]]_0$, rational functions by Laurent series $\tilde{\Phi} = \Omega((t))$, we can even arrange in the above example that A is a *radical domain*. But in these examples the correct (i.e. tight) scalar extension is 1-dimensional, not 2-dimensional, so they do not contradict our goal (the Primitive Form Theorem 5.6 that a simple scalar form V is not radical). In fact the dimension 2 situation is not a problem for *simple* pairs and *tight* extensions.

Remark 5.5 (Dimension 2). *Let V be a simple Jordan pair over its centroid Φ with tight scalar extension $\tilde{V} = V\tilde{\Phi} \simeq \mathcal{V}(J(\tilde{q}, \tilde{1}))$ for \tilde{q} a nondegenerate quadratic form of dimension 2 over $\tilde{\Phi}$, such that \tilde{q} and \tilde{t} are anisotropic on V . Then V has regular (even invertible) elements.*

PROOF. By tightness \tilde{V} is simple, so $\tilde{J} = J(\tilde{q}, 1)$ is too, so by (5.4.1) $\tilde{J} = \tilde{\Omega}^+$, so all U_{x^+} lie in the centroid. Then for all $x^+, x^- \neq 0$ in V the

element $x = (x^+, x^-) \in V$ has $Q_x^2 = (U_{x^+}U_{x^-}, U_{x^-}U_{x^+}) \neq 0$ in $\Gamma(\tilde{V})$, so by (5.2)(iv) x is invertible in V . ■

We are now ready to put the pieces together; our main result is

Primitive Form Theorem 5.6. *If V is a simple Jordan pair over its centroid Φ which is a scalar form of some classical pair \tilde{V} of Clifford type over its centroid, a field $\tilde{\Phi} \supset \Phi$, as in the Clifford List 3.4, then V is primitive.*

PROOF. Assume $\tilde{V} = \tilde{\Phi}V$ is classical (in particular, central-simple) over $\tilde{\Phi}$ of Types (Ia-c), (II), (III) in (3.4). It suffices to prove that $\mathcal{R}ad(V) \neq V$ [so that by simplicity, $\mathcal{R}ad(V) = 0$ and V (being semiprimitive and simple) is in fact primitive]. Assume to the contrary that $\mathcal{R}ad(V) = V$ is entirely radical. Then by (5.2) (v) no local algebra $A_b = V_b^+$ of V can have scalar-regular elements in any extension $\bar{\Phi}$. On the other hand, the local algebras $A = A_b$ are scalar forms of the local algebras $\tilde{A} = (\tilde{A})_b$ of a classical \tilde{V} and by the Local Classical Algebra Theorem 4.5 we know these \tilde{A} are scalar forms of $\bar{A} = J(\bar{q}, \bar{1})$ simple of dimension $\neq 2$ over an algebraically closed field $\bar{\Phi}$, so A too is a scalar form of $J(\bar{q}, \bar{1})$. Then by Lemma 5.4 A has scalar-regular elements in the extension $\bar{\Phi}$, a contradiction. ■

6. Zel'manov polynomial on classical pairs

We now address the question of whether each classical pair of List 3.4 is necessarily of Clifford type by showing that any such pair satisfies the Zel'manov polynomial constructed in [5]. We will avoid dealing with superscripts \pm as much as possible and carry out most calculations in a triple system setting.

We denote by $ST = ST(X)$ the **free special Jordan triple system** on an infinite set X of indeterminates, and we fix $b \in X$. Let $[x, y]^{(b)} := xby - ybx$ denote a commutator in the b -homotope $\mathcal{A}(X)^{(b)}$ of the **free associative algebra** $\mathcal{A}(X)$. We first use an inner derivation of $ST^{(b)}$ to construct a **tetrad eater** a in the algebra $ST^{(b)}$ [26] which is then combined to a composite element c into yet another b -commutator to produce a **Zel'manov polynomial** [5], (a hearty pentad-eater which is also a Clifford HPI). Note that although commutators are not Jordan *elements*, their double-commutators are, and their P -operators are Jordan *operators*:

$$\begin{aligned}
(6.1) \quad \mathcal{D}_{x,y}(z) &:= (V_{x,y}^{(b)} - V_{y,x}^{(b)})z = [[x,y]^{(b)}, z]^{(b)}, \\
P_{[x,y]^{(b)}} &:= P_{\{x,b,y\}} - 2(P_x P_b P_y + P_y P_b P_x), \\
U_{[x,y]^{(b)}}^{(b)} &:= U_{\{x,y\}^{(b)}}^{(b)} - 2(U_x^{(b)} U_y^{(b)} + U_y^{(b)} U_x^{(b)}).
\end{aligned}$$

The original pentad-eater [5, p.176] was constructed as follows, where $f^{(b)}$ for a Jordan algebra polynomial f means f computed in the b -homotope of the Jordan system:

$$\begin{aligned}
(6.2) \quad p_{16}(x, y, z_1, z_2, z_3) &:= [[\mathcal{D}_{x,y}^2(z_1)^2, \mathcal{D}_{x,y}(z_2)], \mathcal{D}_{x,y}(z_3)], \\
a_{31} = a &:= p_{16}^{(b)}(x, y, z_1, z_2, z_3) + p_{16}^{(b)}(x, y, z_1, z_3, z_2) \\
&= [[\mathcal{D}_{x,y}^2(z_1)^{(2,b)}, \mathcal{D}_{x,y}(z_2)]^{(b)}, \mathcal{D}_{x,y}(z_3)]^{(b)} \\
&+ [[\mathcal{D}_{x,y}^2(z_1)^{(2,b)}, \mathcal{D}_{x,y}(z_3)]^{(b)}, \mathcal{D}_{x,y}(z_2)]^{(b)}, \\
c_{69} = c &:= p_{35}^{(b)}(x, y, z_1, z_2, z_3, z_4, z_5, z_6) := [V_{a,z_4}^{(b)}, V_{a,z_5}^{(b)}]z_6, \\
d_{101} = d &:= [a, c]^{(b)} \quad (\text{fictitious element!}), \\
m_{611} &:= P_{P_d P_b d} z_7 := P_d P_b P_d P_b P_d z_7 = U_d^{(b)} U_d^{(b)} P_d z_7, \\
z_{1833} &:= m_{611}^3.
\end{aligned}$$

However, for our present purposes we need a *homotope* PI, and z_{1833} is not entirely a product in the b -homotope: m is not a homotope product with z_7 , and z is not the cube in the b -homotope. It is easy to turn m into a homotope-product by replacing z_7 by $P_b z_7$, but simply replacing z by $P_m^{(b)} m = m^{(3,b)}$ does not clearly produce a hearty eater. The referee has kindly suggested the following modification:

$$\begin{aligned}
(6.2.1) \quad m_{613} = m &:= p_{307}^{(b)}(x, y, z_1, \dots, z_7) := U_{d^{(3,b)}}^{(b)} z_7 := (U_d^{(b)})^3 z_7, \\
m'_{1227} = m' &:= p_{307}^{(b)}(P_b x, P_b y, P_b z_i) \\
&= P_b m'' := P_b(p_{307}^{(P_b b)}(x, y, z_i)), \\
z_{2453} = z &:= P_m m' = U_m^{(b)} m''.
\end{aligned}$$

We must modify the argument of [5, Th. 4.5, p.182] to make sure z remains a hearty pentad eater. The argument of [5, Thm. 4.5 p.175] shows that the

inner ideal $m_{611}(b; x, y, z_1, \dots, z_6, ST) = P_{P_d P_b d} ST \subseteq H_5^{odd}$ consists of *odd hearty eaters* (eat pentads when in an odd position), and the argument of [5, p.182] shows that if h_2 is an odd eater then any $P_x h_2$ is an *even hearty eater*, so for *any* $m = m_{611}(b; x, y, z_i, z_7), m' = m_{611}(b'; x', y', z'_i, z'_7)$ the element $P_m m'$ is both an even and an odd hearty eater, i.e. a hearty pentad eater.

To make sure this is still a b -homotope polynomial, take $b' = P_b c, x' = P_b x, y' = P_b y, z'_i = P_b z_i$; since $J(T^*(c)) \xrightarrow{T} J(c)$ is a homomorphism of Jordan algebras for any structural pair (T, T^*) and any element c , we know in particular that P_b is an algebra homomorphism $ST^{(P_b c)} \rightarrow ST^{(c)}$ (cf. [4, 2.1], [20, Prop. 5.4(iii) p.3057]), and so preserves the polynomial $p_{307} : P_b(m'') := P_b(p_{307}^{(P_b c)}(x, y, z_i)) = p_{307}^{(c)}(P_b(x), P_b(y), P_b(z_i)) = m'$. Here m'' is a b -homotope product: recalling that $J^{(P_b c)} = (J^{(b)})^{(c)}$ is the c -homotope of $J^{(b)}$, any Jordan product in $J^{(P_b c)}$ is a product in $J^{(b)}$ involving the element c (for example, the basic products [5, (0.6) p.161] in $J^{(P_b c)}$ are (i) $U_x^{(P_b c)} = U_x^{(b)} U_c^{(b)}, (ii) x^{(2, P_b c)} = U_x^{(b)} c, (iii) V_{x, y}^{(P_b c)} = V_{x, U_c^{(b)} y}^{(b)}, (iv) V_x^{(P_b c)} = V_{x, c}^{(b)}$). Therefore the hearty pentad eater $z = P_m m' = P_m P_b m'' = U_m^{(b)} m''$ is a b -homotope product for any c , in particular for $c = b$. (In the case of pairs it is more natural to take $b = b^\varepsilon, c = c^{-\varepsilon}$ rather than $c = b = (b^+, b^-)$).

It is easy to check that z does not vanish strictly on $\mathcal{H}_3(\Phi)$ (for $b = 1$ the element m' reduces to m , and the same calculation as [5, p.176] gives $z = m^3 \neq 0$).[†] Thus we have constructed our Zel'manov polynomial.

We can now give a general argument why the pairs on the Clifford List satisfy the Zel'manov polynomial. All our struggling to patch up z_{1833} to make a homotope product z_{2453} was unnecessary: the products already vanish at degree 69, $c = 0$.

Theorem 6.3. (i) *All Classical Jordan Pairs on the Clifford List 3.4 (as well as the BiCayley Pair) satisfy $c \equiv 0$ (hence $z \equiv 0$).*

(ii) *Indeed, whenever all local algebras of a Jordan system J are forms of quadratic form algebras, then J satisfies $m \equiv 0$ (hence the Zel'manov*

[†] Any $a \in HE_4^{(b)}$ would also do. [26, 14.1(i) p.192] used $a = p_{16}^{(b)}(x, y, z_1, z_2, z_2)$, where the substitution of [5, p.176] (modified to take $z_2 = u_{13} + tu_{23}$ for an indeterminate t) yields $a = t[u_{12} + 2t(e_3 - e_1)], c = t^2[u_{13}], d = -t^9(t^2 + 1)[(e_{23} - e_{32}) + 4t(e_{31} - e_{13})], z = -t^{18}(16t^2 + 1)^2[u_{23} - 4tu_{13}]$, but the linearized version $z_2 \rightarrow z_2, z_3$ of (6.2) is much easier to compute with!

polynomial $z \equiv 0$ too).

(iii) More generally, if $a_0(X)$ is any identity in the local algebra at b , $\overline{a_0^{(b)}(X)} \equiv \bar{0}$ in A_b , then

- (6.3.1) for any $c_0 \in U_{a_0}^{(b)}(J)$ (e.g. $c_0 = V_{a_0, z_4}^{(b)} V_{a_0, z_5}^{(b)} z_6$ or $[V_{a_0, z_4}^{(b)}, V_{a_0, z_5}^{(b)}] z_6$, since $V_{a_0, J}^{(b)} V_{a_0, J}^{(b)} J \subset U_{a_0}^{(b)} J$) we have $d_0 := [z_8, c]^{(b)}$, $m_0 := P_{d_0} z_7 = 0$, $z_0 := P_{d_0} z_9 = 0$ for any z_7, z_8, z_9 ;
- (6.3.2) if $U_{a_0}^{(b)}(J) = 0$ (e.g. if $b = e^-$ for a maximal idempotent e in a pair $J = V$, $V_0(e) = 0$) then already $c_0 = 0$ for any $c_0 \in V_{a_0, J}^{(b)} V_{a_0, J}^{(b)} J$.

PROOF. (ii) implies $z = 0$ in (i) since all local algebras A_b are Clifford forms by the Local Classical Algebra Theorem 4.5. (iii) implies (ii) since when the local algebras A_b are quadratic form algebras they satisfy the Clifford HPI

$$(6.3.3) \quad a_0 := a_{15} := p_8^{(b)}(x, y, z_1, z_2, z_3) \\ \text{for } p_8(x, y, z_1, z_2, z_3) := [[\mathcal{D}_{x,y}(z_1)^2, z_2], z_3]$$

since squares of commutators lie in the center in Clifford forms (e.g. [26,7.6]), thus a_0 vanishes in each A_b (so $a_{31} = p_8^{(b)}(x, y, \mathcal{D}_{x,y}(z_1), \mathcal{D}_{x,y}(z_2), \mathcal{D}_{x,y}(z_3)) + p_8^{(b)}(x, y, \mathcal{D}_{x,y}(z_1), \mathcal{D}_{x,y}(z_3), \mathcal{D}_{x,y}(z_2))$ does too).

For (iii), we start from the vanishing of a_0 in $A_b = J^{(b)}/\text{Ker}_J(b)$:

$$a_0 \in \text{Ker}_J(b) : P_b a_0 = P_b P_{a_0} b = 0.$$

These orthogonalities imply others:

$$(6.3.4) \quad P_b a = 0 \quad \implies V_{J,a}^{(b)} = V_{J, U_a^{(b)} J}^{(b)} = 0,$$

$$(6.3.5) \quad P_b P_a b = 0 \quad \implies V_{J, a^{(2,b)}}^{(b)} = 0,$$

$$(6.3.6) \quad P_b a = P_b P_a b = 0 \implies V_{U_a^{(b)} J}^{(b)} = 0,$$

$$(6.3.7) \quad P_b a = P_b P_a b = 0 \implies V_{U_a^{(b)} J, J}^{(b)} = 0.$$

For (4), note $V_{x,a}^{(b)} = L_{x,P_b a} = 0$ and $V_{x,U_a^{(b)}y}^{(b)} = L_{x,P_b(P_a P_b y)} = L_{x,P_b a y} = 0$.

For (5), $V_{x,a^{(2,b)}}^{(b)} = L_{x,P_b(P_a b)} = 0$. For (6), linearizing $x \rightarrow x, 1$ in $V_{x,U_a x} = V_{U_a x, a}$ in the unital hull of the Jordan algebra $J^{(b)}$ gives $V_{U_a x} + V_{x,a^2} = V_{a \circ x, a}$; then $V_{U_a J} \subset V_{J,a} + V_{J,a^2} = 0$ by (4), (5). It is easy to see why (7) holds in special systems: $V_{U^{(b)}(a)x,y}^{(b)}(z) = (abxba)(byb)z + z(byb)(abxba) = abx(bab)ybz + zby(bab)xba = 0$ using only the condition $bab = 0$. To get (7) for general Jordan systems when $\frac{1}{2} \notin \Phi$ we need both parts of the definition of $\text{Ker}_J(b)$: $V_{U_a x, y} = -V_{y, U_a x} + V_{y \circ U_a x}$ where $V_{J, U_a J} = 0$ by (4) and $U_a x \circ J = 0$ by (6).

Going back to (1), we now can see why $V_{a_0, J}^{(b)} V_{a_0, J}^{(b)} \subset U_{a_0}^{(b)} J$: (7) shows that in the unital hull of $J^{(b)}$ we have $V_{a, x} V_{a, y} = V_{U(a)x, y} + U_a U_{x, y} = 0 + U_a U_{x, y} \in U_a U_J$. Now

$$(6.3.8) \quad \begin{aligned} c \in U_a^{(b)} J = P_a P_b J \quad (P_b a = P_b P_a b = 0) \\ \implies L_{c, b} = P_c P_b = P_b P_c = 0 \end{aligned}$$

since $L_{c, b} \in L_{P(a)P(b)J, b} = V_{U^{(b)}(a)J}^{(b)} = 0$ by (6), and $P_c \in P_a P_b P_J P_b P_a$ is killed on either side by P_b since $P_b P_a P_b = P_{P_b a} = 0$.

In the special case, $c \in abJba$ implies $bc = cb = 0$, hence $d_0 = [z_8, c]^{(b)} = z_8 bc - cb z_8 = 0$ and so $P_{d_0} = 0$. Again this holds in the general case: $P_{d_0} = P_{[z_8, c]^{(b)}} = P_{\{z_8, b, c\}} - 2(P_{z_8} P_b P_c + P_c P_b P_{z_8})$ (by (6.1)) = 0 by (8). Therefore any $m_0 = P_{d_0} z_7 = 0$, $z_0 = P_{m_0} z_9 = 0$ and (1) is established.

For (2), note that if $U_{a_0}^{(b)} = P_{a_0} P_b = 0$ then any $c_0 \in V_{a_0, J}^{(b)} V_{a_0, J}^{(b)} J \subset V_{U_{a_0}^{(b)} J, J}^{(b)} + U_{a_0}^{(b)} U_{J, J}^{(b)} = 0$, and the Zel'manov polynomial already vanishes at the second stage. If $b = e^-$ for an idempotent $e = (e^+, e^-)$ in V , then $\text{Ker}_J(b) = V_1^+(e) + V_0^+(e)$, and when $V_0(e) = 0$ this reduces to $\text{Ker}_J(b) = V_1^+$; then $a \in \text{Ker}_J(b) \implies a = a_1^+$ so that $U_a^{(b)}(V^+) = Q_{a_1^+} Q_{e^-}(V_2^+) \subset Q_{V_1} Q_{V_2} V_2 \subset V_0 = 0$ and $U_a^{(b)} = 0$. This completes the proof of (iii) and (ii).

This shows $z \equiv 0$ in (i). But (i) claims that the Zel'manov polynomial already vanishes at the second stage $c_0 = 0$ for *all* the systems on the Clifford list. By the definition (6.3.2) of a, c it suffices to show that $U_{a_0}^{(b)} = Q_{a_0} Q_b = 0$ for a_0 as in (6.3.3) on a dense set of b 's.

I. Rectangular Pairs: $U_{a_0}^{(b)} \equiv 0$. Let V be the pair $\mathcal{M}_{1,q}(\Phi)$ (resp. $\mathcal{M}_{1,q}(\mathcal{Q}), \mathcal{M}_{2,q}(\Phi)$), let R denote the associative algebra Φ (resp. $\mathcal{Q}, M_2(\Phi)$) and set $J := V^{+(b)}$ for some $b \in V^-$ with the operations

$$(6.4) \quad x^{(2,b)} = xbx, \quad U_x^{(b)}y = xbybx, \quad V_{x,y}^{(b)}z = xbybz + zbybx.$$

J is naturally a left R -module (scalar multiplication is the usual matrix product) and $J \xrightarrow{T} R$ defined by $T(x) = xb$ is clearly an R -module homomorphism. In those terms, (6.4) becomes

$$(6.5) \quad \begin{aligned} x^{(2,b)} &= T(x)x, & U_x^{(b)}y &= T(x)T(y)x, \\ V_{x,y}^{(b)}z &= T(x)T(y)z + T(z)T(y)x. \end{aligned}$$

We obtain an associative algebra (the “ b -homotope” of the associative pair V) by setting $A := J$ with product $xy = T(x)y = xby$, in which case $J = A^J$ (as in (1.3.1)) and $A^J \xrightarrow{T} R^J$ is a Jordan homomorphism: $T(U_x^{(b)}y) = T(T(x)T(y)x) = T(x)T(y)T(x) = U_{T(x)}T(y)$ and $T(x^{(2,b)}) = T(T(x)x) = T(x)^2$. Thus applying T to an inner derivation $\mathcal{D}_{x,y}(z)$, we obtain $T(\mathcal{D}_{x,y}(z)) = \mathcal{D}_{T(x),T(y)}T(z) = [[T(x), T(y)], T(z)]$, and so by (6.5)

$$(6.6) \quad T(\mathcal{D}_{x,y}(z)^{(2,b)}) = [[T(x), T(y)], T(z)]^2 \in \Phi 1$$

which in turn implies (by the definition (6.3.3) of a_0)

$$(6.7) \quad T(a_0) = [[T(\mathcal{D}_{x,y}(z_1)^{(2,b)}), T(z_2)], T(z_3)] = 0,$$

so $U_{a_0}^{(b)}y = T(a_0)T(y)a_0$ (by (6.5)) = 0 (by (6.7)), and we have $U_{a_0}^{(b)} \equiv 0$.

Remark 6.8. In the case $V = \mathcal{M}_{1,q}(\Phi)$, the commutativity of $R = \Phi$ makes it easy to see that a_0 already vanishes: here, $\mathcal{D}_{x,y}(z) = T(z)(T(y)x - T(x)y)$ so $T(\mathcal{D}_{x,y}(z)) = T(z)[T(y), T(x)] = 0$ by commutativity and $\mathcal{D}_{x,y}^2(z) = T(\mathcal{D}_{x,y}(z))(T(y)x - T(x)y) = 0$ so $\mathcal{D}_{J,J}^2 = 0$ and $a_0 = 0$.

II. Alternating Pair: $U_{a_0}^{(b)} \equiv 0$. Let $V = \mathcal{V}(A_5(\Phi))$ with product $Q_x y = -xyx$; let $b \in A_5(\Phi)$ and set $J = A_5(\Phi)^{(b)}$ with product $U_x y = Q_x Q_b y$. We will denote $Z = \text{Ker}_{A_5(\Phi)}(b) = \text{Ker}(P_b)$ by (1.12.2).

The element $b^- \in A_5(\Phi)$ has (matrix) rank 0, 2 or 4. By a Zariski density argument (rank is a polynomial function on $A_5(\Phi)$) it is enough to

show that $U_{a_0}^{(b)}$ vanishes in J for b of maximal rank 4. Once again, as in (1.8.3) we identify V with alternating maps between dual 5-dimensional Φ -spaces W, W^* . We recall from the rank 4 case of Proposition 4.3 that we may assume $b = E_{12} - E_{21} + E_{34} - E_{43}$ and $Z = \{X \in A_5(\Phi) \mid X_{ij} = X_{55} = 0, 1 \leq i, j \leq 4\}$ from which we easily see that

$$(6.9) \quad U_{a_0}^{(b)} J \subset Z \cap \Phi E_{55} = 0 \quad \text{for all } a_0 \in Z.$$

Remark 6.10. If $\text{rank}(b) = 2$, we actually get $a_0 \equiv 0$ on J . As in Prop. 4.3, $A_b := J/Z \simeq \Phi$ so we have $\mathcal{D}_{x,y}(z) = 0$ in A_b , hence $d := \mathcal{D}_{x,y}(z) \in Z$. But then in the special system V , $[(\mathcal{D}_{x,y}(z))^{(2,b)}, u]^{(b)} = (dbd)bu - ub(dbd) = 0$ since $bdb \in P_b Z = 0$; thus $a_0 \equiv 0$.

III. Clifford Pairs: $a_0 \equiv 0$. First suppose $V = \mathcal{V}(J(q))$ is full Clifford with product $Q_x y = q(x, y)x - q(x)y$. Set $J := J(q)^{(b)}$ and $t(x) := q(x, b)$ (so $t(b) = 2q(b)$). We first compute expressions for x^2 , $U_x y$, and $\{xyz\}$ [23, 1.5-1.6]: they are respectively

$$\begin{aligned} x^{(2,b)} &= Q_x b = q(x, b)x - q(x)b = t(x)x - q(x)b, \\ U_x y &= Q_x Q_b y \\ &= q(x, [q(b, y)b - q(b)y])x - q(x)[q(b, y)b - q(b)y] \\ &= [t(x)t(y) - q(b)q(x, y)]x - t(y)q(x)b + q(x)q(b)y, \end{aligned}$$

and so

$$\begin{aligned} \{xyz\} &= [t(z)t(y) - q(b)q(z, y)]x + [t(x)t(y) - q(b)q(x, y)]z \\ &\quad - t(y)q(x, z)b + q(x, z)q(b)y. \end{aligned}$$

From those, we easily compute,

$$\begin{aligned} \mathcal{D}_{x,y}(z) &= \{xyz\} - [[t(x)t(z) - q(b)q(z, x)]y \\ &\quad + [t(x)t(y) - q(b)q(x, y)]z - t(x)q(y, z)b + q(b)q(y, z)x] \\ &= [t(y)t(z) - 2q(b)q(y, z)]x - [t(x)t(z) - 2q(b)q(x, z)]y \\ &\quad + [t(x)q(y, z) - t(y)q(x, z)]b. \end{aligned}$$

With this last expression, we remark that inner derivations are traceless:

$$(6.11) \quad t(\mathcal{D}_{x,y}(z)) = q(\mathcal{D}_{x,y}(z), b) = 0$$

since $t(\mathcal{D}_{x,y}(z)) = [t(x)t(y)t(z) - 2q(b)q(y,z)t(x)] - [t(x)t(y)t(z) - 2q(b)q(x,z)t(y)] + [2q(b)t(x)q(y,z) - 2q(b)t(y)q(x,z)] = 0$ (using $t(b) = 2q(b)$); from (6.11), we then obtain

$$(6.12) \quad \begin{aligned} \mathcal{D}_{x,y}(z)^{(2,b)} &= t(\mathcal{D}_{x,y}(z))\mathcal{D}_{x,y}(z) - q(\mathcal{D}_{x,y}(z))b \\ &= -q(\mathcal{D}_{x,y}(z))b \in \Phi b. \end{aligned}$$

Next, we observe that

$$(6.13) \quad \mathcal{D}_{b,w}(u) = 0 :$$

indeed $\mathcal{D}_{b,w}(u) = [t(w)t(u) - 2q(b)q(w,u)]b - [t(b)t(u) - 2q(b)q(b,u)]w + [t(b)q(w,u) - t(w)q(b,u)]b = 0$. Combining (6.12) and (6.13), we get

$$\mathcal{D}_{(\mathcal{D}_{x,y}(z))^{(2,b)},w}u = \mathcal{D}_{-q(\mathcal{D}_{x,y}(z))b,w}u \in \Phi \mathcal{D}_{b,w}u = 0;$$

in particular, a_0 vanishes on $\mathcal{V}(J(q))$.

Every J in (3.4)(III) is a tight scalar form of some $J(\bar{q}, 1)$, and a_0 vanishes on $\mathcal{V}(J(\bar{q})) \cong \mathcal{V}(J(\bar{q}, 1))$ [via $(id, -) \supset \mathcal{V}(J)$].

6.14 BiCayley Pairs: $U_{a_0}^{(b)} \equiv 0$. Our Zel'manov polynomial cannot possibly vanish on an Albert pair (Type (VI)) [since an Albert algebra becomes $H_3(\mathcal{C}, \Omega)$ in some scalar extension]. Although we have included i-speciality in our definition of pair of Clifford type, the BiCayley pairs (Type (V)) also satisfy $c_0 \equiv 0$ and hence $z \equiv 0^\dagger$.

Here $V = \mathcal{V}(J)$ for the Jordan triple $J = M_{1,2}(K) = [K, K]$, for a Cayley algebra K over Φ , under $P_{[a,b]}[c, d] = [a(\bar{c}a) + b(\bar{d}a), a(\bar{c}b) + b(\bar{d}b)]$. It suffices to prove $c_0 \equiv 0$ for the Zariski-dense set of $b \in V^-$ of maximal rank $\dim_\Phi(Q_b V^+) = 8$ and by Petersson's Conjugacy Theorem [27, Th.5, p.701] any such b is conjugate to $b = [1, 0]$. When $b = [1, 0]$ we have $\text{Ker}(b) = [0, K]$, $A_b \simeq K^+ = J(q, 1)$ for an 8-dimensional quadratic form, and $U_{a_0}^{(b)}x = P_{a_0}P_b x \in P_{[0,K]}P_{[1,0]}[K, K] = P_{[0,K]}[\bar{K}, 0] = [0, 0]$. ■

† We would like to thank Robert Korty, an REU at the University of Virginia, for using Maple to reassure us computationally about our arguments.

7. Primitive pairs of Clifford type

In this section and the next, we will show that for Jordan pairs of Clifford type, primitivity and simplicity are equivalent. Recall that V is of *Clifford type* if it satisfies a Clifford HPI. Our first goal is to prove that a primitive pair V with a Clifford HPI must be simple (recall it was established in [3] that primitive quadratic Jordan PI algebras are simple and unital). The argument revolves around Loos' theory of the socle of Jordan pairs [14]. We begin with a general result about the socle of a strongly prime system (algebra, triple or pair). In what follows, the **capacity** of a Jordan triple system J is defined as the capacity of the corresponding Jordan pair $\mathcal{V}(J) = (J, J)$.

Proposition 7.1. (1) *If J is a nondegenerate Jordan system (algebra, triple or pair) and I an ideal with a covering tripotent ($I \subset J_2(e) + J_1(e)$ for $e \in I$) then $J = I \boxplus J_0(e)$; if in addition J is prime and $I \neq 0$, then $J = I$.*
(2) *If J is strongly prime and $\text{Soc}(J) \neq 0$ is a nonzero system of finite capacity, then $J = \text{Soc}(J)$ is simple.*

PROOF. In (1), it suffices to prove the first assertion. By hypothesis $I = I_2 \oplus I_1 \subset J_2 \oplus J_1 \oplus J_0 = J$, and $e \in I \triangleleft J$ implies $I \supset J_2 + J_1$, so $I = J_2 \oplus J_1$ and $J = I \oplus J_0$ is a module direct sum. We must prove that it is a direct sum of systems

$$(7.1.1) \quad P_I J_0 = P_{J_0} I = \{IIJ_0\} = \{IJ_0J_0\} = 0$$

(for algebras we also need $I \circ J_0 = 0$).

In the prime nondegenerate case (which is what we need in this paper) we easily get $J_0 = 0$: $I \neq 0 \Rightarrow 0 = \text{Ann}_J(I)$ (by primeness) = $\{z \in J \mid P_z I = 0\}$ (by nondegeneracy, [24, 1.7(i)]), and already $P_{J_0} I = 0$ by the Peirce relations.

The result $J = I \boxplus J_0$ still holds in the general nondegenerate case, but we have to look more carefully at the Peirce relations: from $P_{J_1} J_2 + \{J_1 J_1 J_0\} \subset I \cap J_0 = 0$ we get

$$(7.1.2) \quad P_{J_1} J_2 = \{J_1 J_1 J_0\} = 0,$$

and by Peirce orthogonality of J_2 and J_0 the conditions (7.1.1) reduce to

$$(7.1.3) \quad P_{J_1} J_0 = 0,$$

$$(7.1.4) \quad \{J_2 J_1 J_0\} = 0,$$

$$(7.1.5) \quad \{J_1 J_0 J_0\} = 0,$$

$$(7.1.6) \quad J_1 \circ J_0 = 0 \quad (\text{in the algebra case}).$$

(7.1.3) vanishes by nondegeneracy: $z_2 = P_{x_1} x_0$ has $P_{z_2} J = P_{x_1} P_{x_0} P_{x_1} J_2 = 0$ by (7.1.2). From this, (7.1.4) also vanishes by nondegeneracy: $z_1 = \{x_2 x_1 x_0\}$ has $P_{z_1} J = (P_{x_2} [P_{x_1} P_{x_0}] + P_{x_0} [P_{x_1} P_{x_2}] + L_{x_0, x_1} P_{x_2} L_{x_1, x_0} - P_{P(x_0)P(x_1)x_2, x_2}) J$ (see [5, 0.7]) $\subset 0 + 0 + L_{x_0, x_1} P_{x_2} L_{x_1, x_0} J - 0$ (by (7.1.3); (7.1.2); (7.1.2)) $= 0$ [since $L_{x_1, x_0} J_2 = 0$ by Peirce orthogonality, $L_{x_1, x_0} J_1 = 0$ by linearized (7.1.3), and $P_{x_2} L_{x_1, x_0} J_0 \subset P_{J_2} J_1 = 0$]. This has (7.1.5) as consequence: $\{x_1 x_0 y_0\} = \{y_0 x_0 \{x_1 e e\}\} = -\{\{x_1 e y_0\} x_0 e\} + \{x_1 e \{y_0 x_0 e\}\} + \{y_0 \{e x_1 x_0\} e\}$ (using [13, JP14]) $= -0 + 0 + \{y_0 \{e x_1 x_0\} e\}$ where $\{e x_1 x_0\} \in \{J_2 J_1 J_0\} = 0$ by (7.1.4). In an algebra, (7.1.6) is always a consequence of (7.1.4) when e is an algebra *idempotent* since $x_1 \circ x_0 = \{e x_1 x_0\}$.

When e is merely a triple *tripotent* we can argue directly ($x_1 = \{e e x_1\} = e^2 \circ x_1$ has $x_1 \circ x_0 = (e^2 \circ x_1) \circ x_0 = \{e^2 x_1 x_0\} + \{x_1 e^2 x_0\} \in \{J_2 J_1 J_0\} + \{J_1 J_2 J_0\} = 0$ by (7.1.4) and orthogonality), or we can replace e by its associate e^4 , which is a true idempotent ($(e^2)^2 = e^4 = e^2$ is idempotent by nondegeneracy since $U_{e^4 - e^2} = U_e U_{e^3 - e} = 0$ by Macdonald's Theorem, and it is an associate of e since $U_{e^2} U_{e^2} = U_{e^3} U_e = U_e U_e$, $V_{e^2, e^2} = V_{e^3, e} = V_{e, e}$ by Macdonald's Theorem again, so e^2 and e determine the same triple Peirce decompositions).

For (2), we apply (1) to $I = Soc(J)$; by finite capacity there is a maximal tripotent e which covers I , $I \cap J_0(e) = 0$, so $J = I \oplus J_0$, and then by primeness and $I \neq 0$ we have $J_0 = 0$, $J = I = Soc(J)$. Now the socle is always a direct sum of simple ideals [14, theorem 2], so by primeness it must reduce to a single simple ideal. \blacksquare

To get a primitive pair V with Clifford HPI to be simple, we first observe that $Soc(V)$ is simple of finite capacity, then we show that we actually have $Soc(V) = V$.

Proposition 7.2. *If V is strongly prime of Clifford type with $Soc(V) \neq 0$, then $Soc(V)$ is simple and has finite capacity ≤ 2 .*

PROOF. If V is strongly prime, so is $Soc(V)$ (see [24]) and hence $Soc(V)$

is simple and inherits the Clifford HPI. By Litoff's theorem for Jordan pairs [14, theorem 3], $Soc(V)$ must have finite capacity ≤ 2 . ■

Our main result is

Theorem 7.3. *If V is a primitive pair of Clifford type, then $V = Soc(V)$ is simple classical of capacity ≤ 2 .*

PROOF. By (1.20) V has nonzero socle, and by (7.2) (primitivity implies strong primeness) $Soc(V)$ is simple of capacity ≤ 2 ; hence, by (7.1), (2) $V = Soc(V)$. ■

8. Simple pairs of Clifford type

In the previous section, we established that primitive pairs of Clifford type are simple (and classical); we now wish to show that simple pairs are primitive (hence themselves classical). This is done in two stages: we first argue that simple pairs are forms of classical ones, then we invoke the Primitive Form Theorem. The first part of the argument requires a scalar extension that inherits semiprimitivity, a condition difficult to confront head-on. But nondegeneracy is automatically inherited by tight extensions; therefore we choose our field of scalars so as to shrink the Jacobson radical into the lower radical. The following radical preliminaries explain how this is done.

Theorem 8.1. (1) *If A is a Jordan algebra satisfying a polynomial identity, then*

$$\mathcal{M}c(A) = \mathcal{L}(A) = \mathcal{SNil}(A)$$

so A is nondegenerate iff it is free of strictly-nil ideals.

(2) *If J is a Jordan triple system or pair satisfying a homotope polynomial identity, then*

$$\mathcal{M}c(J) = \mathcal{L}(J) = \mathcal{PSNil}(J)$$

so J is nondegenerate iff it is free of psnil ideals.

PROOF. (1) is simply [26, 17.1] (in view of the chain (1.14.1)), and (2) follows by applying (1) to all the PI algebras $J^{(y)}$: by radical surgery and $\mathcal{M}c(J) \subset \mathcal{PSNil}(J)$ it suffices to prove J nondegenerate $\Rightarrow J$ is free of psnil ideals, so assume $I \triangleleft J$ is psnil, i.e. all $I^{(y)}$ remain strictly-nil. By (1) we have

$I^{(y)} = \mathcal{M}c(I^{(y)})$, hence $I = \bigcap_{y \in I} \mathcal{M}c(I^{(y)}) = \mathcal{M}c(I)$ (the methods of [9] for algebras easily extend to triples and pairs) $= 0$ (since the ideal I inherits nondegeneracy from J). ■

Over an uncountable field, nilness and strict-nilness are the same.

Theorem 8.2. *Over an uncountable field Φ ,*

- (1) *Nilness and strict nilness coincide, and hence $\mathcal{SNil}(A) = \mathcal{Nil}(A)$ for any Jordan algebra A ,*
- (2) *Proper nilness and strict proper nilness coincide, and hence $\mathcal{PSNil}(J) = \mathcal{PNil}(J)$ for any Jordan triple or pair J .*

PROOF. Note that (2) follows by applying (1) to all homotopes $A = J^{(y)} : J \text{ pnil} \Rightarrow \text{all } J^{(y)} \text{ nil} \Rightarrow \text{all } J^{(y)} \text{ snil} \Rightarrow J \text{ psnil}$. So the only thing to prove is that nilness implies strict nilness. The argument for Jordan algebras is the same as that for associative algebras. Assume A is nil over an uncountable field; to show strict nilness, it suffices to prove $A[t_1, \dots, t_n]$ is nil,

$$(8.3) \quad A \text{ nil over an uncountable field} \implies A[t_1, \dots, t_n] \text{ nil},$$

and by induction it suffices to show nilness is preserved under adjunction of one indeterminate,

$$(8.4) \quad A \text{ nil} \implies A[t] \text{ nil}.$$

So consider any $x(t) = a_0 + a_1 t + \dots + a_n t^n \in A[t]$. For any $\lambda \in \Phi$, the element $x(\lambda) = a_0 + a_1 \lambda + \dots + a_n \lambda^n \in A$ is nilpotent of some index $d(\lambda)$, so $d : \Phi \rightarrow N$ is defined and $\Phi = \bigcup_{i=1}^{\infty} \Phi_i$ where $\Phi_i = \{\lambda \in \Phi \mid d(\lambda) = i\}$. Since Φ is uncountable, one of the countably many Φ_i must be infinite (even uncountable).

Assume Φ_d is such. Then $x(\lambda)^d = 0$ for infinitely many values of λ , in particular for more than dn values; $x(t)^d = \sum_{k=0}^{nd} p_{k,d}(a_0, \dots, a_n) t^k$ is a polynomial of degree $\leq nd$ in t , so if it has more than nd roots in Φ it must be identically zero as a polynomial, and $x(t)$ is nilpotent in $A[t]$ of index $\leq d$. [If $y(t) = \sum_{k=0}^m b_k t^k$ has $y(\lambda) = 0$ for more than m values λ from a field Φ , then $y(t) = 0$ in $A[t]$: for any linear functional $\varphi \in A^*$, $\varphi(y(t)) = \sum_{k=0}^m \beta_k t^k$ is an ordinary polynomial in $\Phi[t]$ ($\beta_k = \varphi(b_k)$) with more than m roots, so $\varphi(y(t)) \equiv 0$ in $\Phi[t]$ and all $\beta_k = 0$. Thus $\varphi(b_k) = 0$ for all $\varphi \in A^*$ and all k , so $b_k = 0$ for all k , and $y(t) = 0$.] ■

The shrinkage of the radical into the nil radical is standard using

Theorem 8.5 (Amitsur’s Resolvent). *If A is a Jordan algebra over a field Φ and $a \in A$ has large resolvent $\rho(a) = \{\lambda \mid \lambda \hat{1} - a \text{ invertible in } \hat{A}\} \supset \{\lambda \neq 0 \mid \hat{1} - \lambda^{-1}a \text{ invertible in } \hat{A}\}$*

$$|\rho(a)| > 1 + \dim_{\Phi}(A),$$

then a is algebraic over Φ . ■

Next, we shrink the Jacobson radical into the properly-nil radical. Recall that given a Jordan system J over a field Φ , we say that Φ is **large** if $|\Phi| > 2 + \dim_{\Phi}(J)$ [10, 4.5.8].

Theorem 8.6. *For any Jordan system J (algebra, triple or pair) over a large field,*

$$\mathcal{R}ad(J) = \mathcal{P}Nil(J),$$

so J is semiprimitive iff it is free of properly-nil ideals.

PROOF. [17, p.3542] We need only prove $\mathcal{R}ad(J) \subset \mathcal{P}Nil(J)$: if $z \in \mathcal{R}ad(J)$ then all $\lambda^{-1}z \in \mathcal{R}ad(J)$ are quasi-invertible in all $J^{(y)}$, so $\hat{1} - \lambda^{-1}z$ is invertible in $\hat{J}^{(y)}$, $|\rho(z)| \supset |\Phi \setminus \{0\}| > (2 + \dim_{\Phi}(J)) - 1$ (by largeness) $= 1 + \dim_{\Phi}(J^{(y)})$, so by Amitsur z is algebraic in $\hat{J}^{(y)}$; but it also remains in the radical of $\hat{J}^{(y)}$, and the only algebraic elements of the radical are nilpotents. Thus z is nilpotent in each $J^{(y)}$, i.e. it is properly-nilpotent. ■

Putting these together we have our main radical shrinkage theorem.

Theorem 8.7. *If J is a Jordan system (algebra, triple or pair) satisfying an HPI over a large uncountable field, then*

$$\mathcal{M}c(J) = \mathcal{R}ad(J)$$

so J is nondegenerate iff it is semiprimitive. ■

Corollary 8.8. *Any simple Jordan system satisfying an HPI over a large uncountable field is primitive.*

PROOF. J is simple hence nondegenerate; by (8.7) above, J is therefore semiprimitive, hence primitive. ■

Remark 8.9. We advocate using “really large” fields, i.e. fields which are large and (in case J has finite dimension) also uncountable. Even better

would be **rich** fields, those Φ which contain a large uncountable algebraically closed subfield Ω (they need not be algebraically closed themselves). These fields not only shrink radicals, but they also remove division algebras:

Proposition 8.10. *The only Jordan division system over a rich field Φ is Φ itself.*

PROOF. If J is a division system then for $c \neq 0$, $A = J^{(c)}$ [resp. $V^{+(c^-)}$ in the case of pairs] is a Jordan division algebra with unit $1^c = c^{-1}$, and it suffices to prove $A = \Phi 1$. But if $a \notin \Phi 1$ then all $\lambda 1 - a$ for $\lambda \in \Omega$ are nonzero, hence invertible, so by (8.5) and largeness of Ω the element a is algebraic over Ω : $p(a) = 0$ for $p(t) \neq 0$ in $\Omega[t]$. But since Ω is algebraically closed, $p(t) = \prod (t - \lambda_i)$ factors into linear factors $\lambda_i \in \Omega$ and $0 = p(a)^2 = U_{a-\lambda_1} \cdots U_{a-\lambda_n} 1$, contradicting that all $U_{a-\lambda_i}$ are invertible. Hence some $\lambda 1 - a = 0$ and $a \in \Phi 1$. ■ ■

We also note in passing that

Proposition 8.11. *If J is a prime nondegenerate Jordan system then it is a tight cover of all its local algebras A_b for $b \neq 0$:*

$$0 \neq I \triangleleft J \implies 0 \neq \bar{I} \triangleleft A_b \quad (b \neq 0).$$

PROOF. If $I \triangleleft J$, then $I \triangleleft J^{(b)} \implies \bar{I} \triangleleft A_b$, hence $\bar{I} = 0 \implies I \subset \text{Ker}(P_b) \implies P_b I = 0 \implies b \in \text{Ann}(I)$ (by nondegeneracy [24, 1.7]) = 0 (by primeness and $I \neq 0$). This contradicts the assumption $b \neq 0$. ■

This is false in general for a non-prime J : if $J = J_1 \boxplus J_2$ and $b = b_1 \oplus 0$, then $\text{Ker}_J(b) = \text{Ker}(b_1) \oplus J_2$, $A_b = (J_1)_{b_1}$ and the nonzero ideal $J_2 \triangleleft J$ has $\bar{J}_2 = 0$ in A_b .

From this we can avoid “proper” nilness (nilness at each b) and get by with nilness at a single b , provided that J is prime.

Proposition 8.12. *Let J be a prime nondegenerate Jordan system. If an ideal I is strictly-nil in a single PI local algebra A_b for $b \neq 0$, then $I = 0$.*

PROOF. If $I \neq 0$ then $0 \neq \bar{I} \triangleleft A_b$ by primeness and (8.11); if \bar{I} is strictly-nil then $\bar{I} \subset \mathcal{SNil}(A_b) = \mathcal{Mc}(A_b)$ (by (8.1) since A_b is PI) = 0 (J nondegenerate $\implies A_b$ nondegenerate), a contradiction. ■

Assume now that V is a simple pair over its centroid Φ . The obstacle to

primitivity is the obstacle to semiprimitivity: for all we know we may have a simple radical pair (such exist for associative algebras by Sasiada's examples, but of course these are not PI). From the above results, we can get free of the radical by having enough scalars.

Theorem 8.13 (Tight Scalar Extension). *Any simple pair V of Clifford type over its centroid Φ is a scalar form of a classical pair: any tight scalar extension $\tilde{V} = V\tilde{\Phi}$ by a rich field $\tilde{\Phi} \supset \Phi$ is again simple but also primitive and retains the Clifford HPI, so \tilde{V} is classical Clifford over its centroid $\tilde{\Omega} \supset \tilde{\Phi}$ (and $\tilde{V} = V\tilde{\Omega}$ is also a tight scalar extension).*

PROOF. By tightness \tilde{V} inherits simplicity as well as nondegeneracy, by scalarness it inherits the HPI (remember all PI's are assumed to hold strictly); since it is nondegenerate over a rich field it is semiprimitive by (8.7), hence primitive with Clifford HPI, so by Theorem 7.3 the pair \tilde{V} is classical of Clifford type. ■

Combining Theorem 8.13 with (7.3) and the Primitive Form Theorem 5.6, we obtain

Theorem 8.14 (Simple Pairs). *Any simple Jordan pair V of Clifford type is primitive hence classical over its centroid Φ . It is therefore isomorphic to one of the pairs of the Clifford List 3.4:*

- (I) *Rectangular Pairs:* $\mathcal{V}(M, R, f)$ for a central simple Artinian Φ -algebra R of degree ≤ 2 , which thus has one of the three forms
 - (Ia) $\mathcal{M}_{1,q}(\Phi)$, $1 \leq q \leq \infty$,
 - (Ib) $\mathcal{M}_{1,q}(\mathcal{Q})$, \mathcal{Q} quaternion division algebra over Φ , $1 \leq q \leq \infty$,
 - (Ic) $\mathcal{M}_{2,q}(\Phi)$, $2 \leq q \leq \infty$,
- (II) *Alternating Pair:* $\mathcal{V}(A_5(\Phi))$,
- (III) *Clifford Pairs:* $\mathcal{V}(J)$ for a central simple Clifford Jordan algebra J (a tight scalar form of some simple $J(\bar{q}, 1)$ for a nondegenerate quadratic form \bar{q} of dimension $\neq 2$ over an algebraically closed extension field $\bar{\Phi}$ of Φ).

In particular, the simple pairs of Clifford type are not merely i-special but are in fact special.

9. Simple triple systems of Clifford type

Instead of building the structure of triples from scratch, we will rely on our knowledge of pairs and look at $\mathcal{V}(J)$. We would first like to point out that, as in the case of pairs, primitive triples of Clifford type are simple. Remember that a triple is of Clifford type if it satisfies a Clifford HPI.

Theorem 9.1. *If J is a primitive Jordan triple system of Clifford type, then J is simple of finite capacity.*

PROOF. Given a primitive Jordan triple J with a Clifford HPI, form $V = \mathcal{V}(J) = (J, J)$. Then $Soc(J) \neq 0$ by (1.20), so $Soc(V) = (Soc(J), Soc(J)) \neq 0$ and by (7.1) we need only prove the Clifford HPI forces $Soc(V)$ to have finite capacity. By [7, Theorem 5.8 p.228] there are two possibilities: either V itself is primitive with Clifford HPI, or $V \cong W \oplus W^{op}$ ($W = V/I$ for $I \triangleleft V$ maximal with respect to $I \cap I^* = 0$) for W primitive with Clifford HPI. In the first case $Soc(V)$ has finite capacity by (7.3). In the second case, $Soc(W) = W$ is again of finite capacity by (7.3). But then $Soc(V)$ also has finite capacity: if $\{e_i\}$ is an infinite family of orthogonal division idempotents in V , then all but a finite number of $\{e_i + I\}$ are zero in W (by finite capacity there), so $e_i \in I$, $i > n$, for some integer n , and dually, $e_j \in I^*$, $j > m$, some integer m , and thus $e_k \in I \cap I^* = 0$ for $k > \max(m, n)$. ■

Remark 9.2. In fact, any primitive HPI Jordan triple system which isn't of hermitian type is simple (and equals its socle by (1.20)). Indeed from Zel'manov's first classification paper [29], we know that a strongly prime Jordan triple J is either i-special or its central closure $\Gamma^{-1}J$ is simple exceptional finite-dimensional over the field $\Gamma^{-1}\Gamma$. Since the i-special systems are either of Hermitian type or Clifford type (by [6]), it suffices by (9.1) to handle the case of an exceptional triple J .

We first observe that for a map T in the centroid Γ of J ,

$$(1) \quad J \text{ is nondegenerate} \implies Ker(T) \triangleleft J$$

[$Ker(T)$ is clearly outer, and if $x \in J$, $z \in Ker(T)$, then $P_z x \in Ker(T)$ since $P_{T(P(z)x)} = T^2 P_z P_x P_z = P_{T(z)} P_x P_z = 0$ forcing $T(P_z x) = 0$ by nondegeneracy], and

$$(2) \quad J \text{ is strongly prime} \implies \text{the centroid acts faithfully on } J$$

[since $T \neq 0 \Rightarrow 0 \neq T(J) \triangleleft J$ and $P_{T(J)}Ker(T) = T^2P_JKer(T) = 0 \Rightarrow Ker(T) = 0$ by (1) and primeness].

Assume now that J is primitive with an HPI and $\Gamma^{-1}J$ is simple finite-dimensional over $\Gamma^{-1}\Gamma$. Some local algebra A_b is primitive with a polynomial identity, hence A_b is a simple unital Jordan algebra [3] and by [4, proof of lemma 3.5] the heart H of J is nonzero. Now always

$$(3) \quad J \text{ semiprime with nonzero heart } H \implies H = P_H H + P_J P_H H$$

since $I = P_H H + P_J P_H H$ is an ideal of J contained in H , and is nonzero by semiprimeness of J , so $H \subset I$ by definition of heart, and $H = I$. Then

$$(4) \quad \begin{aligned} J \text{ strongly prime with heart } H \neq 0 \\ \implies T(H) = H \text{ for all } T \neq 0 \text{ in } \Gamma \end{aligned}$$

since $0 \neq T(H)$ (by (2) and $T \neq 0$) is an ideal of J , so again $H \subset T(H)$ by definition of heart, yet by (3) $T(H) \subset P_H T(H) + P_J P_H T(H) \subset H$, hence $T(H) = H$ ($\Leftrightarrow H = T^{-1}H$).

Thus $0 \neq \Gamma^{-1}H \triangleleft \Gamma^{-1}J$ implies $\Gamma^{-1}H = \Gamma^{-1}J$ by simplicity. In particular $J \subseteq \Gamma^{-1}H$, that is, for any $x \in J$ we have $x = T^{-1}h$ for some $T \in \Gamma, h \in H$, so $x \in T^{-1}(H) = H$; in other words, the minimal ideal is $H = J$, and J is simple. \blacksquare

From here on, we assume that J is a simple Jordan triple system of Clifford type over its centroid Φ . By (1.2.1) the Jordan pair $V = \mathcal{V}(J) = (J, J)$ with exchange involution $(x, y)^* = (y, x)$ is $*$ -simple, and has $J \simeq \mathcal{J}(V, *)$ by (1.2.3). Either V is simple in its own right, or it is not simple but merely $*$ -simple, in which case it is the direct sum $V = W \boxplus W^{op}$ of a simple W and its opposite under the exchange involution [if $0 \neq W$ is a proper ideal of V , then $W + W^*, W \cap W^*$ are $*$ -ideals, hence $W + W^* = V$ and $W \cap W^* = 0$ and $V = W \boxplus W^* \simeq W \boxplus W^{op}, * \simeq ex$].

The case $V = W \boxplus W^{op}$ is easy: by (1.2.4) this just means $J = Pol(W)$ is the polarized triple associated to the simple pair W .

We now examine the case where $V = \mathcal{V}(J)$ is already simple. By our classification (8.14), V is isomorphic to either $\mathcal{V}(M, R, f)$, $\mathcal{V}(A_5(\Phi))$ or $\mathcal{V}(I)$ for a simple Clifford Jordan algebra I (a tight scalar form of some simple $J(\bar{q}, 1)$ for a nondegenerate form \bar{q} of dimension $\neq 2$ over an algebraically closed extension field $\bar{\Phi}$ of Φ), and $J \simeq \mathcal{J}(V, *)$ for some involution $*$. If

$V \simeq \mathcal{V}(M, R, f)$ is rectangular of type $\mathcal{M}_{1,q}(\Delta)$ for $q > 1$, we have a rather precise description for the involutions.

Proposition 9.3. *If $V = (M^+, M^-)$ is of type $\mathcal{M}_{1,q}(\Delta)$ for $\Delta = \Phi$ or \mathcal{Q} (quaternion division algebra) and $q > 1$, then any involution $*$ on V has the form $(x^+, y^-)^* = (T^{-1}(y^-), T(x^+))$ for a Φ -linear bijection $T : M^+ \rightarrow M^-$ which satisfies $T(ax^+) = T(x^+)\bar{a}$ for an involution $-$ on Δ , and the Jordan triple $J = \mathcal{J}(V, *) \simeq (V^+)^{(*)}$ under $P_{x^+}y^+ = Q_{x^+}(y^+)^*$ has the form*

$$J = \mathcal{T}(M, \Delta, h) : P_x y = h(x, y)x$$

for a nondegenerate hermitian form h on the left Δ -space $M = M^+$:

$$h(ax, by) = ah(x, y)\bar{b}, \quad h(x, y) = \overline{h(y, x)}.$$

PROOF. $V^+ = M$ is a left vector space over Δ , and we have a bijection $V^+ = M \xrightarrow{*} V^-$ which can be identified with a subspace of the dual of M via the given bilinear form $f : V^+ \times V^- \rightarrow \Delta$. We claim

$$(1) \quad (ax)^* = x^* \sigma_x(a) \quad (a, \sigma_x(a) \in \Delta, x \in M^+).$$

This is trivial if $x = 0$ (take $\sigma_0(a) = a$). As soon as $x \neq 0$, it has $f(x, M^-) = \Delta$, so as in the proof of Prop. 3.1 (with $e_{11} = 1$) we can find $M^+ = \Delta m_1^+ \oplus M_1^+$, $M^- = m_1^- \Delta \oplus M_1^-$ for $m_1^+ = x$, $f(m_1^+, m_1^-) = 1$, $f(M_1^+, m_1^-) = f(m_1^+, M_1^-) = 0$. Then $(ax)^* = [f(m_1^+, m_1^- a) m_1^+]^* = [Q_{m_1^+}(m_1^- a)]^* = Q_{m_1^+}{}^*(m_1^- a)^* = (m_1^+)^* f((m_1^- a)^*, m_1^+{}^*) = x^* \sigma_x(a)$ where $\sigma_x(a) = f((m_1^- a)^*, x^*)$. Note if $x \neq 0$ then $\sigma_x(a)$ is uniquely determined by (1) [since $x \neq 0 \Rightarrow x^* \neq 0$], and always $\sigma_x(1) = 1$, $\sigma_x(0) = 0$. Clearly $\sigma_x(a) = f((m_1^- a)^*, x^*)$ is Φ -linear in a , and by uniqueness $x^* \sigma_x(ab) = \{(ab)x\}^* = \{a(bx)\}^* = (bx)^* \sigma_{bx}(a) = x^* \sigma_x(b) \sigma_{bx}(a)$ from (1) shows

$$(2) \quad \text{if } x \neq 0 : \quad \sigma_x(ab) = \sigma_x(b) \sigma_{bx}(a), \quad \sigma_x(a+b) = \sigma_x(a) + \sigma_x(b), \\ \sigma_x(\alpha a) = \alpha \sigma_x(a),$$

and $[f(x, m^-)x]^* = x^* \sigma_x(f(x, m^-))$ equals $[Q_x m^-]^* = Q_{x^*}(m^-)^* = x^* f(m^-{}^*, x^*)$ which shows

$$(3) \quad f(m^-{}^*, x^*) = \sigma_x(f(x, m^-))$$

(at least for $x \neq 0$, but it is trivially true if $x = 0$ too). Moreover

$$(4) \quad x^*, y^* \text{ independent in } M^- \implies \sigma_x = \sigma_y$$

since then in particular $x, y, x+y \neq 0$ and $x^* \sigma_x(a) + y^* \sigma_y(a) = (ax)^* + (ay)^* = [a(x+y)]^* = (x+y)^* \sigma_{x+y}(a) = x^* \sigma_{x+y}(a) + y^* \sigma_{x+y}(a)$ forces the coefficient of x^*, y^* on both sides to coincide, $\sigma_x(a) = \sigma_{x+y}(a), \sigma_y(a) = \sigma_{x+y}(a)$, hence $\sigma_x(a) = \sigma_y(a)$.

Thus by our hypothesis that $q > 1$ we have $\dim(M^-) > 1$, and

$$(5) \quad \dim(M^-) > 1 \implies \sigma_x = \sigma \text{ all have a common value}$$

since if x^*, y^* are dependent there is z^* independent of them both, hence by (4) $\sigma_x = \sigma_z = \sigma_y$.

If all σ_x for $x \neq 0$ have a common value σ , then $- := \sigma$

$$(6) \quad \text{is an involution on } \Delta \text{ and } h(x, y) := f(x, y^*) \text{ is a nondegenerate hermitian form on } M^+.$$

Indeed, $h(x, M^+) = f(x, M^{+*}) = f(x, M^-) \neq 0$ if $x \neq 0$ so h is left nondegenerate. It is hermitian: $h(y, x) = f(y, x^*) = \overline{f(m^{-*}, x^*)}$ (for some $m^- \in M^-$ since $*$ maps V^- onto V^+ as well) $= \overline{f(x, m^-)}$ (by (3)) $= \overline{f(x, y^*)}$ (since $m^- = (m^-)^{**} = y^*$) $= \overline{h(x, y)}$. [In particular, this shows h is nondegenerate in the second variable as well.] Certainly h is Δ -linear in the first variable, and it is conjugate-linear in the second since $h(x, ay) = f(x, (ay)^*) = f(x, y^* \bar{a})$ (by (1), (5)) $= f(x, y^*) \bar{a} = h(x, y) \bar{a}$. (This also follows directly from hermitianity once we note that $-$ is an anti-homomorphism.)

(2) shows $-$ is a Φ -linear anti-homomorphism (note $\sigma(ab) = \sigma(b) \sigma(a)$ is trivial if $b = 0$), and it is involutory ($\bar{\bar{a}} = a$): $ah(x, y) = h(ax, y) = \overline{h(y, ax)} = \overline{h(y, x) \bar{a}} = \overline{\bar{a} h(y, x)} = \bar{\bar{a}} h(x, y)$ so choosing $h(x, y) \neq 0$ we get $\bar{\bar{a}} = a$. ■

Remarks 9.4. The result fails for $q = 1$: $V = \mathcal{M}_{1,1}(\Delta) = (\Delta m_1^+, m_1^- \Delta)$ has invertible elements and is obtained by doubling the Jordan algebra Δ^+ . The involutions in this case are all (T, T^{-1}) for T in the structure group of Δ^+ . Here $T = L_a A$ for $a = T(1)$ and A an automorphism of Δ^+ , hence (by Hua's theorem) either an isomorphism or anti-isomorphism of Δ . In the case where Δ has no involution at all (e.g. $\Delta \not\cong \Delta^{op}$), none of the involutions on $\mathcal{M}_{1,1}(\Delta)$ come from involutions on Δ .

A complete description of involutions on rectangular pairs $\mathcal{V}(R, M, f)$ for simple artinian R is given in [25]. ■

In (1.13.2) we saw that if we know $\mathcal{V}(J)$ up to isomorphism, we know J up to isotopy:

$$(9.5) \quad \mathcal{V}(J_1) \simeq \mathcal{V}(J_2) \implies J_1 \simeq J_2 \text{ isotopic.}$$

For example, if $\mathcal{V}(J) \simeq \mathcal{M}_{1,1}(\Phi)$ or $\mathcal{M}_{1,1}(\mathcal{Q})$, then $\mathcal{V}(J)$ is a Clifford pair and J is isotopic to a Clifford triple system built out of a quadratic form. This covers the case left out by (9.3).

More generally, if $\mathcal{V}(J) \simeq \mathcal{V}(I)$ for I a tight scalar form of a Clifford Jordan algebra $J(\bar{q}, 1)$, then $J \simeq I^{(\sigma)}$. Comparing (1.9), (1.10.1) shows $J(\bar{q}, 1) = J(\bar{q})^{(-)}$ so $J(\bar{q}, 1)^{(\sigma)} = J(\bar{q})^{(\tau)}$ for $\tau = - \circ \sigma$ structural but not necessarily symmetric: $\tau^* = \sigma \circ -$.

If $\mathcal{V}(J) \simeq \mathcal{V}(A_5(\Phi))$, then $J \simeq A_5(\Phi)^{(\sigma)}$ under $P_x \sigma(y) = -x\sigma(y)x = x\tau(y)x$ where $\tau = -\sigma$ is just as symmetric and structural as σ .

Gathering the above remarks, we are now in a position to state

Theorem 9.6 (Simple triple systems). *J is a simple Jordan triple system of Clifford type over its centroid Φ iff J satisfies one of the following:*

- (I) $J \simeq \mathcal{J}(\mathcal{V}(J), *)$ where $\mathcal{V}(J)$ is isomorphic to either $\mathcal{M}_{1,q}(\Delta)$ ($\Delta = \Phi$ or \mathcal{Q}) or $\mathcal{M}_{2,q}(\Phi)$. In the first case, J has the form
$$J = \mathcal{T}(M, \Delta, h), \quad P_x y = h(x, y)x$$
for a nondegenerate hermitian form h on the left Δ -space M .
- (II) $J \simeq I^{(\tau)}$ is isotopic to a tight scalar form I of a Clifford Jordan algebra $J(\bar{q}, 1)$ of a nondegenerate quadratic form \bar{q} over an algebraically closed field $\bar{\Phi} \supset \Phi$ where $P_x y = \bar{q}(x, \tau(y))x - \bar{q}(x)\tau(y)$ for an invertible symmetric structural transformation τ on I as in (1.13.1).
- (III) $J \simeq A_5(\Phi)^{(\tau)}$ is isotopic to $A_5(\Phi)$, where $P_x y = x\tau(y)x$ [$\tau^* = \tau \in \text{Str}(A_5(\Phi))$]
- (IV) $J \simeq V^+ \oplus V^- = \text{Pol}(V)$ (c.f. (1.1.1)) where V is one of the simple pairs of Clifford type found in (8.14). ■

10. Strongly prime systems of Clifford type

We refer the reader to [13] for generalities concerning Peirce decompositions. We recall that in a Jordan pair V , two idempotents e and f are **associates** (denoted $e \approx f$) if e and f have the same Peirce spaces, e is **indecomposable** if it can't be written as the sum of two orthogonal idempotents, **primitive** if all its associates are indecomposable (note that this is called *strictly primitive* in [15]), and **local** if $V_2(e)$ is a local Jordan pair.

Certainly, any finite-dimensional nondegenerate pair V contains some

idempotent, since $\mathcal{R}ad(V) = \mathcal{M}c(V) = 0$ by (1.14.2) and by [13, 10.9]:

(10.1) a non-radical pair $V \neq \mathcal{R}ad(V)$ with dcc on principal inner ideals must contain a local idempotent.

Our next result will involve the lifting of idempotents and it is important that this be done orthogonally ([15, Proposition 1], [13, 10.9]):

(10.2) if $I \triangleleft V$ is a pnil ideal, then countable sets of orthogonal idempotents lift orthogonally from V/I to V .

A pair V (resp. triple system J) is said to be **idempotent-finite** if V (resp. $\mathcal{V}(J)$) contains no infinite family of pairwise orthogonal idempotents. Note that for triple systems this is stronger than tripotent-finiteness (tripotents of J are just the *symmetric* idempotents of (J, J)): it amounts to finiteness of “orthogonal” regular elements x_i , where orthogonality brings in the pairing elements y_i such that (x_i, y_i) are pair idempotents. For idempotent-finite systems we have

(10.3) if an idempotent-finite Jordan triple or pair contains an idempotent, then it contains a primitive idempotent

(otherwise given e , some associate of e decomposes orthogonally into nonzero idempotents $e_1 + \tilde{e}_1$, then some associate of \tilde{e}_1 decomposes, and we have $e_1 + e_2 + \tilde{e}_2$, and so on, leading to an infinite orthogonal family).

Our goal in this section is to imbed a strongly prime system J into some ultraproduct the factors of which behave well in the sense of the following theorem (see [29, §3, lemma 18]).

Theorem 10.4. *Let V be a nondegenerate idempotent-finite locally finite-dimensional Jordan pair of Clifford type over a field Φ . Then $V = I_1 \oplus I_2 \oplus \dots \oplus I_m$ is a direct sum of finitely many simple ideals (where the I_i are precisely all the minimal ideals of V). Any maximal orthogonal family of primitive idempotents must have at least m members.*

PROOF. First, V contains at least one nonzero idempotent: by nondegeneracy and (8.1)(2), V is not locally nilpotent, so some finitely-generated

subpair W is not nilpotent; by local finite-dimensionality and (1.14.2), $W \neq \text{Rad}(W)$ is finite-dimensional, hence has a nonzero idempotent by (10.1). Then by (10.3), V contains at least one primitive idempotent; let $\xi = \{e_i\}_{i=1}^n$ be any maximal orthogonal family of primitive idempotents in V (such exist by idempotent-finiteness). Note that ξ is actually maximal among *all* orthogonal families of nonzero idempotents (not just primitive ones): if $e \neq 0$ were orthogonal to ξ , idempotent-finiteness of $V_2(e)$ would guarantee by (10.3) a primitive $e_0 \in V_2(e)$, which would automatically still be orthogonal to $\xi \subset V_0(e)$, and $\xi' = \{e_0, e_1, \dots, e_n\}$ would be an orthogonal family of primitive idempotents, contrary to maximality of ξ .

Define a partial ordering $i \prec j$ and an equivalence relation $i \sim j$ on the indices $\{1, \dots, n\}$ based on the ideals $Id_V(e_i)$ generated by the e_i 's in V :

$$(10.5) \quad \begin{aligned} i \prec j &\text{ iff } Id_V(e_i) \subset Id_V(e_j) \text{ (equivalently } e_i \in Id_V(e_j)), \\ i \sim j &\text{ iff } Id_V(e_i) = Id_V(e_j). \end{aligned}$$

[Note that we haven't yet established that the $Id_V(e_i)$'s are simple, so we don't know containment $Id_V(e_i) \subset Id_V(e_j)$ implies equality, so we must distinguish the two relations.] Partition the indices $\{1, \dots, n\} = N_1 \cup N_2 \cup \dots \cup N_m$ into equivalence classes with respect to \sim . Note that automatically $n \geq m$. We pick $i_k \in N_k$, $1 \leq k \leq m$, and we single out e_{i_1}, \dots, e_{i_m} ; renumbering if necessary, we may assume these to be e_1, \dots, e_m . We claim first that the corresponding ideals span V :

$$(10.6) \quad V = I_1 + I_2 + \dots + I_m, \quad I_k := Id_V(e_k), \quad 1 \leq k \leq m.$$

Set $I := I_1 + \dots + I_m$. Note that all $e_j \in Id_V(e_j) \subset I_k \subset I$, so $e = (e^+, e^-)$, $e^\varepsilon := \sum_{k=1}^n e_k^\varepsilon$ lies in I , and the Peirce spaces $V_2^\varepsilon(e) + V_1^\varepsilon(e) = Q_{e^\varepsilon}V + \{e^\varepsilon e^{-\varepsilon} V^\varepsilon\} \subset I$ so $V = I + V_0(e)$ and $V_0 := V_0(e)$ has no primitive idempotent (by maximality of the e_i 's) hence no idempotents at all (by (10.3)); thus by (10.1) and local-finiteness, finitely-generated subsystems of V_0 are all radical and V_0 is locally-nilpotent, i.e. $V_0 = \mathcal{L}(V_0) = \mathcal{M}c(V_0) = 0$ (by (8.1 (2)), since V_0 has a Clifford HPI, and by Peirce inheritance of nondegeneracy [13, 5.10]) and $V = I$.

We wish to improve on (10.6) and prove that the sum is actually direct (and that the I_k 's are the minimal ideals of V). We remark that if the indices $i \prec j$ as in (10.5), then $e_i \in Id_V(e_j) = \mathcal{M}(V)e_j + \mathcal{M}(V)Q_{e_j}V$ (where $\mathcal{M}(V)$ stands for the multiplication algebra of the system V), that is $e_i = M_{ij}(e_j)$

is a “multiple” of e_j . (Here a **multiple** of x is a sum of terms $W(x)$ and $WQ_x w$ for multiplication operators W and elements w .) We let

$$(10.7) \quad B_0 := \langle e_1^\pm, \dots, e_n^\pm, \text{ and all elements involved} \\ \text{in the multiples } M_{ij} \text{ when } i \prec j \rangle \subset V.$$

Thus e_i can be “connected” to e_j within B_0 . The following lemma shows how it suffices to concentrate our efforts on certain finitely-generated subpairs.

Lemma 10.8. *In order to prove that the sum $V = I_1 + \dots + I_m$ is direct and the I_k 's are the minimal ideals of V , it suffices to show that for every finitely-generated subpair $B \supset B_0$, we have*

$$(10.9) \quad \overline{B} := B/\mathcal{L}(B) = \overline{B \cap I_1} \oplus \dots \oplus \overline{B \cap I_m}, \\ \text{where } \overline{B \cap I_k} \text{ are the minimal ideals of } \overline{B}.$$

PROOF. Note that each $\overline{B \cap I_k}$ is nonzero since $B \cap I_k \supset B_0 \cap Id_V(e_k) \ni e_k$ and $e_k \notin \mathcal{L}(B)$. To show the directness of the sum, suppose $x_1 + \dots + x_m = 0$ in V for $x_i \in I_i$; for any finitely-generated $B \supset B_0 := \langle B_0, x_1, \dots, x_m \rangle$ we have $\overline{x_1} + \dots + \overline{x_m} = \overline{0}$ in \overline{B} with $\overline{x_i} \in \overline{B \cap I_i}$, so $\overline{x_i} = 0$ and $x_i \in \mathcal{L}(B)$ by directness in (10.9), $1 \leq i \leq m$. To show each x_i vanishes it suffices to show each x_i lies in $\mathcal{L}(V) = \mathcal{M}c(V) = 0$ by (8.1 (2) and nondegeneracy). But this will follow from

$$(10.10) \quad x \in \mathcal{L}(V) \iff x \in \mathcal{L}(B) \text{ for all finitely generated} \\ \text{subpairs } B \text{ of } V \text{ containing a fixed } B_{00} \ni x.$$

The implication \implies is easy; for the converse, if x lies in all $\mathcal{L}(B)$ then $Id_V(x)$ is locally-nilpotent since any $\langle y_1, \dots, y_r \rangle \subset Id_V(x)$ is nilpotent: if we write $y_j = M_j(x)$ as a multiple of x and let

$$B = \langle B_{00} \text{ and all elements in the multipliers } M_j \rangle,$$

then all $y_j \in Id_B(x)$ (BY CONSTRUCTION OF B) $\subset \mathcal{L}(B)$ (since $x \in \mathcal{L}(B)$ BY HYPOTHESIS) imply $\langle y_1, \dots, y_r \rangle \subset \mathcal{L}(B)$ is nilpotent. This establishes (10.10) hence directness.

Always in a semiprime direct sum, the pieces are simple iff they are minimal, in which case they are *all* the minimal ideals. Indeed, a simple

ideal is always minimal, a minimal and nontrivial direct summand is always simple, and if I is a minimal ideal which involves the simple direct summand I_k ($\pi_k(I) \neq 0$) then $\pi_k(I) = I_k$ by simplicity, so $I_k = Q_{I_k} I_k$ (simplicity) = $Q_{I_k} \pi_k(I) = Q_{I_k} I$ (directness) $\subset I$ implies I equals I_k by minimality.

For minimality of each $I := I_k = Id_V(e_k) := Id(e)$, let $0 \neq L \triangleleft V$ such that $L \subset I$; by nondegeneracy and (8.1) (2), $\mathcal{L}(V) = 0$, so L cannot be locally nilpotent, and there exists a non-nilpotent $\langle x_1, \dots, x_n \rangle \subset L$. Then $\langle x_1, \dots, x_n \rangle \not\subset \mathcal{L}(B)$ for $B = \langle B_0, x_1, \dots, x_n \rangle \supset B_0$, $0 \neq \langle x_1, \dots, x_n \rangle \subset \overline{L \cap B}$ forces $\overline{L \cap B} = \overline{I \cap B}$ (by minimality in (10.9)) $\Rightarrow \bar{e} \in \overline{L \cap B} \Rightarrow e = x + z$ for some $x \in L \cap B$ and $z \in \mathcal{L}(B)$. But then $e \in L \cap B$ (hence $I = Id(e) \subset L$ so $I = L$ is minimal) by the

Lemma 10.11. *If e is an idempotent in a Jordan pair V of the form $e = x + z$ where*

- (1) z is nilpotent in $V^{(e)}$ (e.g. $e \in B$ and z is properly-nil in B , or $z \in \mathcal{L}(B)$),
- (2) x lies in some $C = (C^+, C^-)$ with $(C^\varepsilon)^{(e^{-\varepsilon})}$ a subalgebra of $(V^\varepsilon)^{(e^{-\varepsilon})}$ and $Q_e Q_e C \subset C$,

then $e \in C$.

PROOF. We have $e = Q_e Q_e e = y + w$ for $y = Q_e Q_e x$ in $C \cap V_2(e)$ [by $Q_e Q_e C \subset C$] and $w = Q_e Q_e z \in V_2(e)$ still nilpotent in $V^{(e)}$ [by the Shifting Principle for $T = Q_e Q_e$ we see $w^{(n,e)} = T(z)^{(n,e)} = T(z^{(n,T^*(e))}) = T(z^{(n,e)})$ eventually vanishes when z is nilpotent in $V^{(e)}$]. This moves us into the unital algebra $\mathcal{A} = V_2^{(e)}$: $y, w \in \mathcal{A}$ with $e = 1$ so $0 = w^n = (1 - y)^n = 1 + \sum_{k=1}^n \binom{n}{k} (-1)^k y^k$ and $e = 1 = -\sum_{k=1}^n \binom{n}{k} (-1)^k y^k \in C$. ■ ■

We now proceed to establish (10.9). Let B be an arbitrary finitely generated (hence finite-dimensional since V is assumed locally-finite-dimensional in (10.4)) subsystem containing B_0 and form $\overline{B} = B/\mathcal{L}(B)$ ($\mathcal{L}(B) = \mathcal{R}ad(B)$ by (1.14.2)). Then \overline{B} is a direct sum of a finite number of simple ideals $\overline{B}_1 \oplus \dots \oplus \overline{B}_s$ and \bar{e}_i is at least an indecomposable idempotent in \overline{B} : indeed, $\overline{B}_2(\bar{e}_i) = \overline{B}_2(e_i)$ is nondegenerate, finite-dimensional (hence finite capacity), so if $\bar{e}_i = \bar{f} + \bar{f}'$ decomposes into nonzero orthogonal idempotents in $\overline{B}_2(\bar{e}_i)$ we can by (10.2) lift to orthogonal $f, f' \in B_2(e_i)$ so $g := f + f'$ in $B_2(e_i)$ has $\bar{g} = \bar{e}_i$, hence g is invertible in $B_2(e_i)$ (by [15, Lemma 1]), so $g \approx e_i$ is an

associate of e_i , which would contradict primitivity of e_i in V (remember that primitivity means indecomposability of all associates). In particular, each \bar{e}_i must lie entirely in one of the \bar{B}_k 's, $1 \leq k \leq s$, in which case $\bar{B}_k = Id_{\bar{B}}(\bar{e}_i)$ by simplicity of \bar{B}_k . [If $\bar{e}_i = \bar{f}_1 \oplus \dots \oplus \bar{f}_s$ then by directness each \bar{f}_k is an idempotent, so by indecomposability only one \bar{f}_k appears.]

Each \bar{B}_j must contain at least one \bar{e}_i [if some \bar{B}_j didn't, directness of the sum would provide an idempotent in \bar{B}_j orthogonal to the maximal family $\bar{\xi}$, maximality which is ensured by that of ξ among *all* orthogonal families and the fact (10.2) that idempotents lift orthogonally]. Finally, we can show that $i \prec j$ implies $i \sim j$:

$$\begin{aligned}
(10.12) \quad i \sim j &\iff i \prec j \\
&\iff \bar{e}_i \text{ and } \bar{e}_j \text{ lie in the same } \bar{B}_k, \quad 1 \leq k \leq s \\
&\iff Id_{\bar{B}}(\bar{e}_i) = Id_{\bar{B}}(\bar{e}_j) = \bar{B}_k.
\end{aligned}$$

Clearly $i \sim j \Rightarrow i \prec j$ by (10.5). To see $i \prec j \Rightarrow \bar{e}_i, \bar{e}_j$ lie in the same \bar{B}_k , recall $e_i = M_{ij}(e_j)$ where M_{ij} is a B_0 multiplication by definition (10.7), hence $\bar{e}_i = \bar{M}_{ij}(\bar{e}_j)$ where \bar{M}_{ij} is a \bar{B} multiplication, so if \bar{e}_j lies in $\bar{B}_k \triangleleft \bar{B}$ so does \bar{e}_i . Clearly $\bar{e}_i, \bar{e}_j \in \bar{B}_k \Leftrightarrow Id_{\bar{B}}(\bar{e}_i) = Id_{\bar{B}}(\bar{e}_j) = \bar{B}_k$ by simplicity of \bar{B}_k . To show $Id_{\bar{B}}(\bar{e}_i) = Id_{\bar{B}}(\bar{e}_j) \Rightarrow i \sim j$, it suffices to show $Id_{\bar{B}}(\bar{e}_i) \subset Id_{\bar{B}}(\bar{e}_j) \Rightarrow i \prec j$ (since by (10.5) $i \sim j \Leftrightarrow i \prec j$ and $j \prec i$); but when $\bar{e}_i \in Id_{\bar{B}}(\bar{e}_j)$ then $\bar{e}_i = \bar{N}_{ij}(\bar{e}_j)$ is a \bar{B} -multiple of \bar{e}_j and $z_i := e_i - N_{ij}(e_j) \in \mathcal{L}(B)$. Then $e_i = x_i + z_i$ for $z_i \in \mathcal{L}(B)$ and $x_i := N_{ij}(e_j) \in Id_V(e_j)$, so by Lemma (10.11) we see $e_i \in Id_V(e_j)$ and so $i \prec j$.

Thus the equivalence classes N_k are in 1-1 correspondence with the \bar{B}_k , and we may assume the numbering is chosen so that

$$(10.13) \quad i \in N_k \iff \bar{e}_i \in \bar{B}_k \quad (k = 1, 2, \dots, m = s).$$

To complete the proof of 10.9 we must show the simple minimal ideals \bar{B}_k are precisely the $\bar{B} \cap \bar{I}_k$. But $\bar{B} \cap \bar{I}_k = \bar{B} \cap Id_V(e_k) \supset Id_B(e_k) = Id_{\bar{B}}(\bar{e}_k) = \bar{B}_k$. To show conversely that $\bar{B} \cap \bar{I}_k \subset \bar{B}_k$ we need to show that $b = M_V(e_k) \in \bar{B} \cap \bar{I}_k = \bar{B} \cap Id_V(e_k)$ has \bar{b} entirely contained in \bar{B}_k . The difficulty is that M_V is a multiplication from V , not necessarily from B , so we don't immediately know $\bar{b} = \bar{M}_V(\bar{e}_k) \in \bar{B}_k$, only that $\bar{b} = \bar{b}_1 \oplus \dots \oplus \bar{b}_m$ for $\bar{b}_i \in \bar{B}_i$. Instead we argue as follows. By simplicity of \bar{B}_i , we have $\bar{b}_i \neq 0 \implies Id_{\bar{B}}(\bar{b}_i) = \bar{B}_i \ni \bar{e}_i$ (recall our renumbering), so $\bar{e}_i = M_{\bar{B}}(\bar{b}_i)$ for

some \overline{B} -multiplication, $\overline{e}_i = Q_{\overline{e}_i} Q_{\overline{e}_i} M_{\overline{B}}(\overline{b}_i) = Q_{\overline{e}_i} Q_{\overline{e}_i} M_{\overline{B}}(\overline{b}) = \overline{Q_{e_i} Q_{e_i} M_B(b)}$ (all other $Q_{\overline{e}_i} Q_{\overline{e}_i} M_{\overline{B}}(\overline{b}_j)$ for $j \neq i$ fall in $Q_{\overline{e}_i} Q_{\overline{e}_i} \overline{B}_j = 0$ by orthogonality $Q_{\overline{B}_i} \overline{B}_j = 0$). Thus $e_i = Q_{e_i} Q_{e_i} e_i = x + z$ for $z \in \mathcal{L}(B)$ nilpotent in $V^{(e_i)}$, $x = Q_{e_i} Q_{e_i} M_B(b) \in C := Id_B(b)$. Applying (10.11) shows $e_i \in C = Id_B(b) = Id_B(M_V(e_k)) \subset Id_V(e_k)$, hence by our original definition (10.5) $i \prec k$, so $i \sim k$ by (10.12), and since e_i, e_k are equivalence class representatives we must have $i = k$. Thus $\overline{b}_i = 0$ for $i \neq k$ and \overline{b} reduces to $\overline{b} = \overline{b}_k \in \overline{B}_k$. This ends the proof of (10.9) and with it the proof of Theorem 10.4. ■

We are now ready to obtain the structure of strongly prime Jordan triples and pairs. We formulate the triple case in terms of $*$ -pairs (pairs with involution), where primeness or simplicity for a $*$ -pair means $*$ -primeness (no orthogonal $*$ -ideals) or $*$ -simplicity (nontriviality and no proper $*$ -ideals).

Theorem 10.14. *A prime nondegenerate Jordan pair V (resp. $*$ -pair, triple) of Clifford type over Φ is a Φ -form of a simple Jordan pair (resp. $*$ -pair, triple) of Clifford type.*

PROOF. The first step is to pass from Φ to a large uncountable Φ -field $F \supset \Phi Id_V$, i.e. $|F| > N$ where V is spanned over Φ by N elements. We form the tensor extension $V_F = F \otimes_{\Phi} V$ and then tighten it to get a tight scalar extension \overline{V} ($V \subset \overline{V}$ with $FV = \overline{V}$). By *tightness* \overline{V} is a prime nondegenerate pair (resp. $*$ -pair), by *scalar extension* \overline{V} remains of Clifford type (it strictly satisfies the same Clifford HPI), and F remains just as large for \overline{V} (if V is spanned over Φ by N elements, they also span \overline{V} over F). If we can prove that \overline{V} is a scalar form of a simple pair (resp. $*$ -pair) $\overline{\overline{V}}$, then so is V ($\overline{\overline{V}} = \overline{\overline{F}} \overline{\overline{V}} = \overline{\overline{F}} \overline{\overline{F}} V = \overline{\overline{F}} V$). Thus we may replace Φ, V by F, \overline{V} from the start and *assume that $\Phi = F$ is a large uncountable field*. Note that by (8.7) and (1.14.2) the $*$ -invariant radicals $\mathcal{M}c, \mathcal{L}$ and $\mathcal{R}ad$ all coincide on such V , so V is nondegenerate $\Leftrightarrow \mathcal{L} = 0 \Leftrightarrow V$ is semiprimitive.

The second step is to imbed V in a “finite environment” (having local-finite-dimensionality and idempotent-finiteness as required for (10.4)). We noted V is semiprimitive, so $V \cong \prod_{\alpha \in S} V_{\alpha}$ for $V_{\alpha} := V/I_{\alpha}$ primitive over its centroid $F_{\alpha} \supseteq F$ for ideals I_{α} with $\bigcap_{\alpha \in S} I_{\alpha} = 0$. [For $*$ -pairs the I_{α} need not be $*$ -ideals, but the $L_{\alpha} = I_{\alpha} \cap I_{\alpha}^*$ are $*$ -ideals and $W_{\alpha} := V/L_{\alpha} \cong V_{\alpha} \boxplus V_{\alpha}^*$ is a $*$ -pair imbedded in the direct sum of V_{α} and a copy of its opposite under the exchange involution, so W_{α} is $*$ -primitive.] By (7.3) each V_{α}

(hence its dual V_α^*) is *simple classical* of Clifford type, in particular by (3.5) has *bound 2* on families of orthogonal idempotents and is *locally d -bounded-dimensional* over F_α . [Being imbedded in the direct sum of 2 such pairs, W_α remains of Clifford type but now has *bound 4* on idempotents and is *locally $2d$ -bounded-dimensional* over F_α (where a centroidal $\gamma \in F_\alpha$ acts via (γ, γ^*) on $W_\alpha \subset V_\alpha \boxplus V_\alpha^*$ for $\gamma^* = * \circ \gamma \circ *$.)]

Thus each V_α [resp. W_α] is “finite”, but because the index set S is usually infinite the product $\prod_{\alpha \in S} V_\alpha$ does not have idempotent-finiteness, and has locally d -bounded dimension only over $\prod_{\alpha \in S} F_\alpha$, which is not a field. We must shrink the direct product using an ultrafilter. By subdirectness, the support set $\mathcal{F}_a = \{\alpha \in S \mid \pi_\alpha(a) \neq 0\}$ for $\pi_\alpha : V \rightarrow V_\alpha$ [resp. $\pi_\alpha : V \rightarrow W_\alpha$] is nonempty for $a \neq 0$ in V , and by primeness is directed downwards. [For the case of $*$ -pairs, $*$ -primeness guarantees $a, b \neq 0$ generate nonzero $*$ -ideals I, K with $Q_I K \neq 0$, so any $c \neq 0$ in $Q_I K$ has $\pi_\alpha(c) \neq 0 \Rightarrow \pi_\alpha(Q_I K) \neq 0 \Rightarrow \pi_\alpha(I), \pi_\alpha(K) \neq 0 \Rightarrow \pi_\alpha(a), \pi_\alpha(b) \neq 0$ in W_α , therefore $\mathcal{F}_c \subset \mathcal{F}_a \cap \mathcal{F}_b$.] Thus $\{\mathcal{F}_a \mid a \neq 0 \text{ in } V\}$ can be enlarged to an ultrafilter \mathcal{F} , with V imbedded in the ultraproduct pair $\tilde{V} = (\prod_\alpha V_\alpha)/\mathcal{F}$ [resp. $*$ -pair $(\prod_\alpha W_\alpha)/\mathcal{F}$] over the field $\tilde{F} = (\prod_\alpha F_\alpha)/\mathcal{F}$ containing F . The ultraproduct \tilde{V} is nondegenerate since each V_α, W_α is, but we don't know \tilde{V} remains prime [resp. $*$ -prime]. [Once the structure theory is complete - which is of no help to us in this paper- we know by [2, Theorem 1.10, p.28] that there is an elemental condition for strong primeness so the ultraproducts do inherit it.] Since ultraproducts inherit all elementary properties common to all factors, we at least know \tilde{V} is of Clifford type with *bound 2* [resp. 4] on orthogonal idempotents and is locally d -[resp. $2d$]-dimensional over the field \tilde{F} . [Note having fixed bounds $d, 2d$ is crucial here: ultraproducts need not inherit mere idempotent-finiteness or local-finite-dimensionality from their factors, witness the subsystem generated by a single $z = \prod_{n=1}^\infty z_n$ for z_n nilpotent of index n . \tilde{V} inherits the fixed Clifford HPI from V , or by (6.3) and (7.3) we know it inherits the Zel'manov polynomial identity from all the primitive V_α .] We may as well discard \tilde{V} and keep only the subpair $\tilde{F}V$. This inherits *Clifford type, bound 2* [resp. 4] on idempotents, and *local d -[resp. $2d$]-bounded-dimensionality*, and $\tilde{F} \supset F$ remains large (since $\tilde{F}V$ is again spanned by N elements).

So far we have sacrificed everything (primeness and nondegeneracy) for finiteness of $\tilde{F}V$. We can restore nondegeneracy, at least by radical surgery: the quotient $\bar{V} = \tilde{F}V/\mathcal{L}(\tilde{F}V)$ is a nondegenerate pair [resp. $*$ -pair, since \mathcal{L} is

always $*$ -invariant] (since $\mathcal{L}(\bar{V}) = 0$ by radical surgery, hence $\mathcal{M}c(\bar{V}) = 0$ too using largeness of \tilde{F} , (1.14.2), and (8.7)). It is still a *scalar extension* of V (note V remains imbedded $V \xrightarrow{\varphi} \bar{V}$ since $V \cap \mathcal{L}(\tilde{F}V) \subset \mathcal{L}(V) = \mathcal{M}c(V) = 0$ by (1.14.2) and (8.7) again). Any quotient automatically inherits the *Clifford HPI* and *local d -[resp. $2d$]-bounded-dimensionality*, and because $\mathcal{L}(\tilde{F}V)$ is pnll we know the quotient inherits the *bound 2* [resp. 4] on idempotents by (10.2). Thus we have all the ingredients necessary for (10.4), and $\bar{V} = I_1 \boxplus \dots \boxplus I_m$ is a finite direct sum of its (simple) minimal ideals.

[In the $*$ -prime case, the induced involution on \bar{V} must send minimal ideals to minimal ideals, so either $I_k^* = I_k$ (and $S_k := I_k$ is itself a simple $*$ -pair), or $I_k^* \neq I_k$ and $S_k := I_k \boxplus I_k^*$ is a $*$ -simple but not simple $*$ -pair). Thus $\bar{V} = S_1 \boxplus \dots \boxplus S_r$ is a direct sum of $*$ -simple pairs S_k .]

Finally, we restore primeness by showing that V imbeds into one of the simple pieces I_k or S_k as a form via $V \xrightarrow{\pi_k \circ \varphi} I_k$ (π_k the projection onto I_k or S_k). Here, $K_k := \text{Ker}(\pi_k \circ \varphi)$ are ideals [resp. $*$ -ideals] of V with $\cap_k K_k = 0$ since φ is an imbedding. But by *primeness* of V its nonzero ideals [resp. $*$ -ideals] satisfy the finite intersection property, so some K_k is already 0, and $\pi_k \circ \varphi$ is injective. Thus $V \xrightarrow{\pi_k \circ \varphi} I_k$ [resp. S_k] with $\tilde{F}\pi_k(V) = \pi_k(\tilde{F}V) = \pi_k(\bar{V}) = I_k$ [resp. S_k]. We have shown V is a scalar form of the simple pair I_k [resp. $*$ -pair S_k], and the theorem is established for pairs and $*$ -pairs.

Since the category of triples is equivalent to that of $*$ -pairs, the theorem for triples follows immediately. In more detail, if J is a strongly prime triple then $V = \mathcal{V}(J) = (J, J)$ is a strongly $*$ -prime $*$ -pair, hence by the above is a scalar form of a $*$ -simple $*$ -pair $\bar{V} \simeq (\bar{J}, \bar{J}) = \mathcal{V}(\bar{J})$ for a triple \bar{J} which is a scalar extension $\bar{J} = \tilde{F}J$ of J . If \bar{V} is actually simple, then \bar{J} is simple (with “tight cover” \bar{V}); if \bar{V} is merely $*$ -simple, then $\bar{V} = \bar{W} \boxplus \bar{W}^*$ is a direct sum of a simple \bar{W} and its opposite under an exchange involution, so by (1.2.4) $\bar{J} = \text{Pol}(\bar{W})$ is the polarized triple of the simple pair \bar{W} (with “loose cover” \bar{V}), hence by (1.1.2) \bar{J} is simple. Thus J is a form of a simple triple \bar{J} of Clifford type. This completes the proof of the theorem. \blacksquare

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